MA 5219 - Logic and Foundations of Mathematics 1
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogic.html`

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Write up one homework not taken by classmates in the Forum and present it in the tutorial. Check the post titles for seeing whether a homework is already written up and put the homework number into the post title when you write up a homework. Homeworks 2... are for Week 2, Homeworks 3... are for Week 3 and so on.

2.1 Boolean Formulas. Let the function $maj(x, y, z)$ output that value which is taken up from at least two of the three inputs and let $neg(x)$ map x to $1 - x$, that is, $neg(0) = 1$ and $neg(1) = 0$. Assume that Boolean constants $0, 1$ are given. New functions can be constructed by concatenating old functions and replacing inputs either by input-variables or constants. Provide definitions for the functions $x \wedge y$ (AND) and $x \vee y$ (OR) using maj, neg and $0, 1$. Assume two inputs x, y.

2.2 Majority and Minority. Define $maj(x, y, z)$ returning the more frequent value of the three inputs and $mino(x, y, z)$ returning the less frequent value of the three inputs using $\wedge, \vee, \neg$ and brackets. So $maj(1, 1, 0)$ and $maj(1, 1, 1)$ are both 1 and $mino(1, 1, 0)$ and $mino(1, 1, 1)$ are both 0.

2.3 Power of Minority. Show that $\wedge$, $\vee$ and $\neg$ can be defined using $mino$ and Boolean constants $0, 1$.

2.4 No Constants. Can the tasks from Homework 2.1 and 2.3 be done without explicitly using 0 and 1? If yes, explain how to do it, if not, explain why it cannot be done. Assume that two inputs x, y are used.

2.5 Exclusive Or. The function $EOR(x_1, \ldots, x_n)$ is 1 if and only if an odd number of the inputs is 1 which might be either variables or constants or other terms. If it are only two terms, one can either write $EOR(x_1, x_2)$ or $x_1 \oplus x_2$. Show the following:
(a) There is a rule which says that $A \vee B \vee C = EOR(A, B, C, A \wedge B, A \wedge C, B \wedge C, A \wedge B \wedge C)$. Prove that rule by case distinction over the number of subfunctions A, B, C which are 1. Furthermore, explain how this rule generalises to other number of inputs, say give the formula for the cases of $A, A \vee B, A \vee B \vee C \vee D$.
(b) Every Boolean function can be represented as an EOR over terms which are the conjunctions of some, possibly negated, inputs where the empty EOR takes the value 1.

2.6 More on EOR I. Recall that CNF is one big conjunction over clauses (disjunctions of literals) and a DNF is one big disjunction over conjunctive terms. Replace $(A \wedge B \wedge C) \oplus (\neg A \vee \neg B)$ by terms in CNF and DNF which use as few terms (disjunction terms connected by a conjunction and conjunctive terms connected by a disjunction) as possible.

2.7 More on EOR II. Replace $(A \wedge B \wedge C) \oplus (\neg A \wedge (B \oplus C))$ by terms in CNF and DNF which use as few terms (disjunction terms connected by a conjunction and conjunctive terms connected by a disjunction) as possible.

2.8 Implication I. Define $\wedge$ and $\vee$ and $\leftrightarrow$ using $\neg$ and $\rightarrow$ where $A \rightarrow B$ is 1 if $A = 0$ or $B = 1$; $A \leftrightarrow B$ is 1 if $A = B$. Here note that the brackets in $A \rightarrow B \rightarrow C$ are usually placed as $(A \rightarrow (B \rightarrow C))$. So if one wants $(A \rightarrow B) \rightarrow C$, this pair of brackets has to be put explicitly.

2.9 Implication II. Define maj and $mino$ using $\neg$ and $\rightarrow$.

2.10 Complexity of formulas I. Let $F(x_1, x_2, \ldots, x_n) = 1$ if and only if the number of variables which are 1 is a multiple of 3. Assume that $n = 4$. Write the corresponding DNF explicitly. Furthermore, determine the number of conjunctive terms for $n = 3, 4, 5, 6, 7$. Here the number for $n = 2$ is 1, namely the term $x_1 \wedge x_2$.

2.11 Complexity of formulas II. Let $F(x_1, x_2, \ldots, x_5) = 1$ if the binary input number is a prime number. Compute a formula with as few $\wedge$ and $\oplus$ as possible. The constants 0 and 1 can be used. Do not use other connectives.

2.12 Complexity of formulas III. Let $F(x_1, x_2, \ldots, x_5) = 1$ if the binary input number is a square number. Compute a formula with as few $\wedge$ and $\oplus$ as possible. The constants 0 and 1 can be used. Do not use other connectives.

3.1 Worlds. Recall that a world is an entity which assigns a truth-value to every atom – alternatively, one can also take W as the set of all formulas which are true in the corresponding world. These two views can be translated into each other.

$W \models A$ means that the world makes the formula A true and $W \models X$ means that the world makes all formulas in a set X of formulas true. $X \models A$ means that every world which makes all formulas in X true, also makes the formula A true. Let p, q, r be atoms.

Now consider the formulas $p \rightarrow \neg q$, $q \rightarrow \neg r$, $r \rightarrow \neg p$. How many worlds are there which make these formulas simulatneously true? Here two worlds count as one if they have the same behaviour on p, q, r.

3.2 Two Worlds I. Let V and W be two different worlds, let $X = \{A :$ exactly one of $V \models A$ and $W \models A$ holds$\}$, let $Y = \{A :$ both of $V \models A$ and $W \models A$ hold$\}$. Show the following.
(a) There is an atom p_k such that either $(V \models p_k$ and $W \models \neg p_k)$ or $(V \models \neg p_k$ and $W \models p_k)$.
(b) If $A, B \in Y$ then $A \wedge B \in Y$.

3.3 Two Worlds II. Let V, W, X, Y as in 3.2. Show the following.
(c) There are formulas $A, B \in X$ with $A \vee B \notin X$.
(d) If $A, B, C \in X$ then at least one of the formulas $A \vee B$, $A \vee C$, $B \vee C$ is in X.

3.4 Two Worlds II. Let V, W, X, Y as in 3.2. Show the following.
(e) If $A \in Y$ then there exist $B \in X$ and $C \in X$ with $\emptyset \models (A \leftrightarrow B \vee C)$.
(f) If $A \wedge B, A \wedge \neg B$ are both in X then $A \in Y$.

3.5 Two Worlds II. Let V, W, X, Y as in 3.2. Show the following.

(g) Assume $(A \to B) \wedge (B \to C) \in Y$. Can one choose A, B, C such that $A \notin X \cup Y, B \in X$ and $C \in Y$?
(h) Assume $(A \to B) \wedge (B \to C) \in Y$. Can one choose A, B, C such that $A \in Y$, $B \in X$ and $C \notin X \cup Y$?
(Note that this homework was updated to make the formula meet the intention, what was not the case before.)

3.6 Logical Implication I. Assume that there are infinitely many atoms and that $W \models X$ for all worlds W which make only finitely many atoms true. Show that $W \models X$ for all worlds W and that X contains only tautologies.

3.7 Logical Implication II. Construct a set X of formulas such that $W \models X$ is true iff W makes at most two atoms true and all others false. Note that X can be infinite.

3.8 Logical Implication III. Assume that X is a set of formulas and $Y = \{A \vee B : A, B \in X \text{ and } A \neq B\}$. Depending on the choice of X, which of the following cases can occur:
(a) X and Y have the same logical consequences.
(b) There is a formula C such that $X \models C$ and $Y \not\models C$.
(c) There is a formula D such that $Y \models D$ and $X \not\models D$.
Provide examples of the corresponding X and Y for those cases which can be satisfied.

3.9 Tautologies I. (a) Show that there an infinite sequence $A_1, A_2, \ldots$ such that for all n, $A_n \to A_{n+1}$ is a tautology and there are worlds V and W such that $V \models A_n$ and $W \models \neg A_n$.
(b) What happens if one furthermore postulates that for every world U there is an n with $U \models A_n$? Does there such a sequence $A_1, A_2, \ldots$ of formulas exist?

3.10 Tautologies II. Assume that $X = \{(A \to B) \to A : A, B \text{ are formulas}\}$ and $Y = \{(A \to B) \to B : A, B \text{ are formulas}\}$. Are all formulas in X tautologies? Are all formulas in Y tautologies?

4.1 Implication Proof System I. A proof system consists of a schema which has place-holders $A, B, C, \ldots$ which can be replaced by any propositional formula. Furthermore, one has certain rules how to apply the derivation of a formula. For example, if only $\to$ is used and nothing else, one can derive many formulas using the following axiom schemas: $A \to (B \to A)$, $(A \to B \to C) \to (A \to B) \to (A \to C)$, $((A \to B) \to A) \to A$. The rule used is just this one: If $A \to B$ and A are derived, one can also derive B (modus ponens). Use these rules to prove the following: $A \to B \to B$.

4.2 Implication Proof System II. Use the schema from 4.1 to prove $A \to A$. Hint, start with $A \to B \to A$ and $B = (A \to A)$.

4.3 Implication Proof System III. Use the schema from 4.1 to show that one can prove $p \to r$ from the set $X = \{p \to q, q \to r\}$; when proving some formula using a set X, one can also use besides the rule given in 4.1 the rule to put A whenever $A \in X$.

4.4 Implication and Negation I. Enhance the proof system of 4.1 by the rules $\neg A \to A \to B$ and $(A \to B) \to (\neg A \to B) \to B$ and $(A \to B) \to (\neg B \to \neg A)$. Now prove $(\neg A \rightarrow A) \rightarrow A$. You can use the proof from 4.2 without reproducing it.

4.5 Implication and Negation II. Use the rules from 4.4 to make a proof for B from the set $\{\neg p, p\}$ where p is some atom.

4.6 Proof System for Exclusive Or. Assume that only the connective $\oplus$ is used to build the formulas in X inductively from logical constants $0, 1$ and atoms $p_0, p_1, \ldots$. For the ease of notation, it is assumed that formulas are written without brackets, as the operation $\oplus$ is associative. Is there a set of rules which is permits to prove all formulas of a finite set of exclusive or formulas Y from a set of exclusive or formulas X whenever $X \models Y$? If so, give the set of rules; if not, explain why such a set of rules cannot exist.

4.7 Axiom Sets I. The following homeworks should describe a structure by a set of axioms. First make a set of axioms for a structure $(U, +)$ such that the so obtained group is the same as the set of logic values $\{0, 1\}$ together with the operation $\oplus$ (exclusive or). Besides $+$, also equality $=$ and variables v, w, x, y, z and quantifiers $\forall, \exists$ can be used, variables range always over U.

4.8 Axiom Sets II. Do the same as in 4.7, but for the semigroup of the structure $(U, +)$ with $U = \{0, 1, 2\}$ and $x + y = \max\{x, y\}$.

4.9 Axiom Sets III. Consider the following set of axioms: $\forall x\, \forall y\, [x + y = y + x]$, $\forall x\, \forall y\, \forall z\, [(x + y) + z = x + (y + z)]$, $\exists x\, \forall y\, [x + y = x]$. Find a three-element set for this structure which is not isomorphic to the structure from homework 4.8.

4.10 Axiom Sets IV. Provide an axiom set for the set $\{0, 1/2, 1\}$ of the truth-values in three-valued fuzzy logic and $\wedge$, $\vee$ as operations on these values. Here these operations coincide with min and max of the two input truth values. Furthermore, if A takes the value q then $\neg A$ takes the value $1 - q$. Which of the rules for deduction in logic with $\wedge$ and $\vee$ and $\neg$ are sound in this logic, that is, do always make valid conclusions from the previous formulas.

5.1 Zorn's Lemma I. Assume that one has only some of the axioms of set theory, say that every two sets have a union and an intersection and that $B \subseteq C$ if and only if $B \cup C = C$. Note that it is also true that $B \subseteq C$ if and only if $B \cap C = B$. Furthermore, assume that Zorn's Lemma is true.
Show that the following is true: Let A be a nonempty set of sets. Show that there is a largest set C such that $C \subseteq B$ for all $B \in A$. That is, use Zorn's Lemma to show that there is a maximal such set C and if there is a set D with $D \subseteq B$ for all $B \in A$ then $D \subseteq C$. Note that C coincides with the operation $\bigcap A$ in set theory, so the infinite intersection operator exists in this type of set theory.

5.2 Zorn's Lemma II. Under the assumptions laid out in 5.1, assume that A is a set of subsets of a set E. Use Zorn's Lemma to show that there is a set C such that $C = \bigcup A$, that is, C is the union of all sets B with $B \in A$.

5.3 Properties of Order-Relations I. Given a set S with an order-relation R on S and that every two elements x, y satisfy xRy or yRx or both of these. Show that every such pair (S, R) is reflexive, that is, satisfies xRx for all $x \in S$. Furthermore, show that if (S, R) is symmetric then xRy holds for all $x, y \in S$.

5.4 Properties of Order-Relations II. Consider the relation xRy iff for all prime numbers p, p divides x if and only if p divides y. Let $S = \{2, 3, 4, \ldots\}$ be the natural numbers from 2 onwards.
(a) Is this relation R reflexive? Prove your answer.
(b) Is this relation R symmetric? Prove your answer.
(c) Is this relation R transitive? Prove your answer.
(d) Is this relation R antisymmetric? Prove your answer.
Note that here a relation R is transitive if xRy and yRz implies xRz. A relation R is symmetric if xRy implies yRx. A relation R is antisymmetric if xRy and yRx imply $x = y$.

5.5 Symmetry of relations. For the notions of "reflexive", "symmetric" and "antisymmetric" from 5.4 and the set $S = \{0, 1, 2\}$, provide a relation R on S which satisfies all three properties.

5.6 Irreflexive relations. Let S be a nonempty set. A relation R is called irreflexive if and only if there is no $x \in S$ with xRx. Is there a relation R on S which is irreflexive, symmetric and transitive? If so, provide an example. If not, prove why such a relation does not exist. For this S can be chosen as desired.

5.7 Finite groups and order-relations I. Let $G_k = \{0, 1, \ldots, k-1\}$ and for $x, y \in G$, let $x + y$ be addition modulo k, that is, if in the natural numbers $x + y \geq k$ then one assigns the value $x + y - k$ in G_k. Furthermore, let $k \geq 3$ and xRy if and only if $x + 1 = y$ or $x + 2 = y$ in G_k. For which $k \geq 3$ does it hold that $(G_k, +)$ is irreflexive and symmetric?

5.8 Finite groups and order-relations II. Let G_k, R as in 5.7 and $k \geq 3$. For which k does it hold that the order-relation R is irreflexive, antisymmetric and connex? Here connex means that all $x, y \in G_k$ satisfy either $x = y$ or xRy or yRx or several of these conditions at the same time.

5.9 Finite groups and order-relations III. Assume on G_k is a relation R which satisfies xRy if and only if $x+1Ry+1$. Is there any such R on G_2 which is irreflexive, antisymmetric and connex? What about G_3 and G_4?

5.10 Properties of logical implication. Recall that $X \models Y$ iff for every $A \in Y$, X logically implies A. Here X, Y are sets of propositional formulas as investigated in Chapter I. Which of the following properties are satisfied by this relation: reflexive, transitive, connex?

5.11 Properties of logical implication. Recall that $X \models Y$ iff for every $A \in Y$, X logically implies A. Here X, Y are sets of propositional formulas as investigated in Chapter I. Which of the following properties are satisfied by this relation: Does this ordering have a least and a greatest element? That is, sets X, Y such that all sets Z of formulas satisfy $X \models Z$ and $Z \models Y$? If yes, provide concrete examples for X and Y; if not, say why they do not exist.

6.1 Models I. Let m, n be natural numbers. For which combinations of m, n are the structures $(m\mathbb{Z}, +, \cdot), (n\mathbb{Z}, +, \cdot)$ the same and for which combinations are they different? If they are different, provide a formula which proves that they are different, these formulas can use integer constants like $-1, 0, 1, m, n$; if they are the same, then say why. However, m, n are not in the logical language but only thought as place holders for the corresponding terms. For example if $m = -1$ then $m \cdot 3$ stands for $(-1 + -1 + -1)$. Superfluous brackets can be omitted, but the outside brackets for m, n are part of these in the formula so that $m \cdot 4$ really means $m + m + m + m$ and nothing else.

6.2 Models II. Consider the structure $(\mathbb{Z}, <, +, -, 0, 1, c)$ where c is a constant symbol and not a term. Now let X be the set of all formulas $1 < c, 1 + 1 < c, 1 + 1 + 1 < c, \ldots$; show that every finite subset Y of X has a model isomorphic to the above one (except for the value of c which might change), but not X itself. Note that X has also some model (by the compactness theorem), but it cannot have $\mathbb{Z}$ as the domain.

6.3 Models III. In what cases do the structures $S_m = (\mathbb{Q} + \sqrt{m} \cdot \mathbb{Q}, 0, 1, \cdot, +, -)$ and $S_n = (\mathbb{Q} + \sqrt{n} \cdot \mathbb{Q}, 0, 1, \cdot, +, -)$ coincide? Here m, n are positive integers. Here all integers are available as terms, let 2 abbreviate $1 + 1$, 3 abbreviate $1 + 1 + 1$ and m, n abbreviate the terms representing their values. Give reasons for the answer.

6.4 Models IV. Consider the case $n = 2$. Provide a formula A such that S_2 is the smallest model which satisfies this formula A and which is a superset of $\mathbb{Q}$.

6.5 Models V. Consider the case $n = 3$. Provide a formula B such that S_3 is the smallest model which satisfies this formula B and which is a superset of $\mathbb{Q}$.

6.6 Models VI. Let $A_n = \forall x \exists y \, [y \cdot n = x]$ where $n \geq 1$ and $y \cdot 1$ stands for y, $y \cdot 2$ stands for $y + y$, $y \cdot 3$ stands for $y + y + y$ and so on. Let $(X_n, +, -, <, 0, 1)$ be the smallest model satisfying A_n and containing $\mathbb{Z}$ as a subset (with the same addition on this subset) such that the domain X_n satisfies $\mathbb{Z} \subseteq X_n \subseteq \mathbb{Q}$. For which n, m does $X_n = X_m$ hold?

6.7 Fields I. Provide a set X of the axioms of a field with constants $0, 1$ for the neutral elements of addition and multiplication, respectively. Note that $0 \neq 1$ is part of the field axioms. Then let Y be X augmented by the axiom $\forall x \, [x = 0 \vee x = 1 \vee x = 1 + 1 \vee x = 1 + 1 + 1 \vee x = 1 + 1 + 1 + 1 \vee x = 1 + 1 + 1 + 1 + 1]$. How many structures satisfy the axioms in Y (up to isomorphism)? Explain the answer.

6.8 Fields II. Explain why one must postulate the communativity of multiplication in the field axioms by providing a structure which satisfies all other field axioms but not this one.

6.9 Groups I. List all groups with four elements by the corresponding tables (up to isomorphism) and provide axioms only satisfied by the listed structures but not by any other ones.

6.10 Groups II. Consider all invertible 3×3 matrices over the finite field of size 2; the group operation is the matrix multiplication. Is there a set of axioms which describes this structure up to isomorphism?

6.11 Groups III. Provide a set of axioms which is satisfied by all infinite groups but not by finite ones. The axiom set is infnite.

6.12 Groups IV. Show if all finite groups satisfy some set X of axioms then also some infinite group satisfies it.

6.13 Groups V. An ordered Abelian group is a group with an order relation $<$ such that $x + y = y + x$ holds for all x, y and $<$ is a linear order and $0 < 1$ and the group satisfies

$$\forall x\, \forall y\, \forall z\, [x < y \leftrightarrow x + z < y + z].$$

Are there finite ordered Abelian groups? Are there infinite ordered Abelian groups? Explain the answers.

7.1 Axioms of Integral Domains. Provide a finite set of axioms for the structure $(U, +, \cdot, 0, 1)$ such that all structures are integral domains, that is, $(U, +)$ is an additive and commutative group with neutral element 0, there is a law of distributivity, multiplication is associative, 1 is neutral element for multiplication and no product of nonzero elements is zero. Search for additional properties which enforce infinity and are first-order definable and are consistent, that is, there is also a model for the combination of all axioms.

7.2 Terms I. Determine the value of the following terms where first x is substituted by $y + 2$ and then y is substituted by 15.
(a) $(x + 3) \cdot (y + 3)$.
(b) $x + 7 \cdot y + 5^5$.
(c) $(x - 1) \cdot (y + 1)$.
(d) $y - x$.

7.3 Terms II. Determine the value of the following terms where first x is substituted by $y + 5$ and then y is substituted by 23.
(a) $(x + 3) \cdot (y + 3)$.
(b) $x + 7 \cdot y + 5^5$.
(c) $(x - 1) \cdot (y + 1)$.
(d) $y - x$.

7.4 Variable Occurrences I. Determine which variables occur bound and which occur free, some might occur in both forms.
(a) $\forall x\, \forall y\, [x \cdot x \neq -1] \wedge (y = 8)$.
(b) $\forall x\, \exists x\, [x = 0] \wedge (y \neq 0 \oplus y = x)$.
(c) $\forall z\, \forall y\, \exists x\, [y + x = z] \vee z = 0$.
(d) $x + y = z + 3$.

7.5 Variable Occurrences II. Determine which variables occur bound and which occur free, some might occur in both forms.
(a) $\forall x\, \forall y\, [y \cdot x \neq -1] \wedge (x = 8)$.
(b) $\forall x\, \exists x\, [1 = 0] \oplus y \neq 0 \oplus y = x$.
(c) $\forall z\, \forall y\, \exists x\, [y - x = z] \wedge z = 0$.
(d) $x + y + z = 3$.

7.6 Formula Truth I. Assume that the underlying structure is the group of integers with the number constants taking the usual values. The defaults of the variables are $x = 5$, $y = 8$, $z = 10$. Determine which formulas evaluate to true and which evaluate to false.
(a) $\forall x\, \forall y\, [x \cdot x \neq -1] \wedge (y = 8)$.
(b) $\forall x\, \exists x\, [x = 0] \wedge (y \neq 0 \oplus y = x)$.
(c) $\forall z\, \forall y\, \exists x\, [y + x = z] \wedge z = 0$.

(d) $x + y = z + 3$.

7.7 Formula Truth II. Assume that the underlying structure is the group of integers with the number constants taking the usual values. The defaults of the variables are $x = 5$, $y = 8$, $z = 10$. Determine which formulas evaluate to true and which evaluate to false.
(a) $\forall x\, \forall y\, [y \cdot x \neq -1] \wedge (x = 8)$.
(b) $\forall x\, \exists x\, [1 = 0] \oplus y \neq 0 \oplus y = x$.
(c) $\forall z\, \forall y\, \exists x\, [y - x = z] \oplus z = 0$.
(d) $(x + y + z) \cdot (x + y + z) = 529$.

7.8 Substitutions I. Determine the outcome of a substitution of a variable x by a term t in the following formulas or say that the subsitution is not allowed and why. Here u, v, w, x, y, z are distinct variables.
(a) $\forall y \exists z\, [x = y + z]$ and $t = y + y$;
(b) $\forall y \exists z\, [x = y + z]$ and $t = u + u$;
(c) $\exists x\, [x = y] \vee x = z$ and $t = y + y$;
(d) $\exists x\, [x = y] \vee x = z$ and $t = u + u$.

7.9 Substitutions II. Determine the outcome of a substitution of a variable x by a term t in the following formulas or say that the subsitution is not allowed and why. Here u, v, w, x, y, z are distinct variables.
(a) $\forall v \exists w\, [x = u + v + w]$ and $t = y + y$;
(b) $\forall v \exists w\, [x = u + v + w]$ and $t = u + u$;
(c) $\exists u\, [x = u + y] \vee x = y + z$ and $t = y + y$;
(d) $\exists u\, [x = u + y] \vee x = y + z$ and $t = u + u$.

7.10 Substitutions III. Determine the outcome of a substitution of a variable x by a term t in the following formulas or say that the subsitution is not allowed and why. Here u, v, w, x, y, z are distinct variables.
(a) $\forall v \exists w \forall y\, [v + w + x = v + w + y]$ and $t = y + y$;
(b) $\forall v \exists w \forall y\, [v + w + x = v + w + y]$ and $t = u + u$;
(c) $\exists u\, [u = x] \vee x = y + z$ and $t = y + y$;
(d) $\exists u\, [u = x] \vee x = y + z$ and $t = u + u$.

7.11 Substitutions IV. Assume the formula $f(x) = -x^2 + y$ is given and one makes a substitution of y by $(x + 5) \cdot (x - 1)$.
(a) What is the degree of the resulting polynomial?
(b) Can one do the same with the formula $\forall x\, [f(x) = -x^2 + y]$? If so, what is the degree?
(c) Is there a function which is equal to a polynomial in several variables such that substituting one of the variables by a further polynomial results in a function not being a polynomial?

7.12 Substitution V. Is there a formula $\alpha(x, y)$ such that $\exists x\, \exists y\, [\alpha(x, y)]$ is true but one can find two terms s, t such that when one substitutes x, y by s, t, respectively, then the formula is false?

7.13 Substitution VI. One has the formula $\forall x\, [f(x) = u \cdot x + v + w]$. Now one can substitute each of u, v, w by $0, 1, 2$ or 3. How many different functions can be created by this method, provided that the underlying structure where the number of functions are counted are the integers.

8.1 Truth in Structures I. Which of the following sentences (= fully quantified formulas) are true in the structure of natural numbers with order and addition?
(a) $\forall x\, \forall y\, [x < y \leftrightarrow \exists z\, [y = x + z + 1]]$.
(b) $\forall x\, \exists y\, [(x = y + y) \oplus (x = y + y + 1)]$.
(c) $\exists x\, \forall y\, [x \neq y + y + y]$.
(d) $\forall x\, \exists y\, [x + y = 0]$.

8.2 Truth in Structures II. Which of the following sentences are true in the structure of all integers (zero and positive and negative numbers)?
(a) $\forall x\, \forall y\, [x < y \leftrightarrow \exists z\, [y = x + z + 1]]$.
(b) $\forall x\, \exists y\, [(x = y + y) \oplus (x = y + y + 1)]$.
(c) $\exists x\, \forall y\, [x \neq y + y + y]$.
(d) $\forall x\, \exists y\, [x + y = 0]$.

8.3 Truth in Structures III. Which of the following sentences are true in the structure of rational numbers?
(a) $\forall x\, \forall y\, [x < y \leftrightarrow \exists z\, [y = x + z + 1]]$.
(b) $\forall x\, \exists y\, [(x = y + y) \oplus (x = y + y + 1)]$.
(c) $\exists x\, \forall y\, [x \neq y + y + y]$.
(d) $\forall x\, \exists y\, [x + y = 0]$.

8.4 Truth in Structures IV. Find two finite monoids with neutral element 0 such that both have at least three distinct elements $0, 1, 2$ and the first monoid satisfies all formulas and the second monoid satisfies none of the formulas.
(a) $\forall x\, \exists y\, [x + x = 1 + y + y + y]$.
(b) $\forall x\, \exists y\, [(x = y + y + 1) \oplus (x = y + y + 2)]$.
(c) $\exists x\, \forall y\, [x \neq y + y + y + y + y]$.
(d) $\forall x\, \exists y\, [x + y = 0]$.

8.5 Truth in Structures V. Find structures $(\{0, 1\}, +)$ in which $+$ is associative and commutative such that these (1) make all three conditions (a,b,c) true, (2) make (a,b) true but not (c), (3) make only (b) true but not (a,c), (4) make none of the conditions true, (5) make only (a,c) true and (b) false. For each such combination, describe the corresponding structure, that is, provide the definition of $+$ to be used on the set $\{0, 1\}$.
(a) $\forall x\, \exists y\, [x + y = 0]$.
(b) $\forall x\, [x + x + x = x]$.
(c) $\exists x\, \forall y\, [x + y \neq 1]$.

8.6 First-Order Sentences I. In logic, a sentence is a formula without free occurrences of variables. Consider the operation given as $x \circ y = y$. List out in first-order logic the following properties it satisfies:
(a) The structure is associative.
(b) If the structure has at least two elements then it is not commutative.
(c) For every x, z one can find an y such that $x \circ y = z$.
(d) If the structure has at least two elements then there are y, z such that there is no x with $x \circ y = z$.

8.7 First-Order Sentences II. Consider the structure $(\mathbb{Z}, 0, 1, +, \cdot, <)$ and recall that p is a prime number if and only if $p > 1$ and whenever $q, r > 1$ then $q \cdot r \neq p$. Provide a formula which says "there are infinitely many prime numbers".

8.8 First-Order Sentences III. For the same integer structure as in 8.7, provide a formula which says that for all $x > 1$, either x is a square or there are infinitely many pairs (y, z) such that $y^2 - x \cdot z^2 = 1$. Note that there is no single symbol in the language to denote the function $y \mapsto y^2$, so one has to find a way to express this with the existing symbols.

8.9 First-Order Sentences IV. For the same integer structure as in 8.7, provide a formula which says that $x \leq y$ if and only if y is the sum of x and four squares of integers.

8.10 First-Order Sentences V. For the same integer structure as in 8.7, provide a formula which says that for all $x > 9$ either x or $x+1$ is neither a square nor a cube nor a fifth power of an integer.

8.11 Logical Implication I. Prove that if $X \models \forall x\,[\alpha]$ and the substitution x by t is allowed in α giving formula β then $X \models \beta$.

8.12 Logical Implication II. Assume that y is neither occurring free in X nor occurs as a variable in α and let β be obtained by substituting x by y. Then $X \models \beta$ implies $X \models \forall x\,[\alpha]$.

8.13 Logical Implication III. Assume that the formula α does not contain any quantifier. Let $\alpha(x)$ be a formula with free variable x and $\alpha(s)$ and $\alpha(t)$ be the formulas obtained by replacing all occurrences of x in α by s and t, respectively. Prove that if $X \models s = t$ and $X \models \alpha(s)$ then $X \models \alpha(t)$.

8.14 Logical Implication IV. Prove that if $X \models \alpha$ then $X \models \exists x\,[\alpha]$.

8.15 Logical Implication V. Provide a set of formulas X and a formula α such that $X \models \alpha$ is true and $X \models \forall x\,[\alpha]$ is false.

8.16 More on First-Order Sentences. For the same integer structure as in 8.7, provide a formula which says that there are infinitely many numbers x such that both $x-1$ and $x+1$ are prime numbers. It is unknown whether this statement is true. (This homework had the same number as an other homework and is therefore moved to the end for Week 8.)

9.1 Famous formulas I. Consider the formula about integers with $+$ and $\cdot$ given as

$$x \geq 0 \Leftrightarrow \exists u\, \exists v\, \exists w\, [x = u \cdot u + v \cdot v + w \cdot w \vee x = u \cdot u + v \cdot v + w \cdot w + 1].$$

Say in words what this formula says. Also determine by consulting Mathematics books or Wikipedia or otherwise, whether the formula is known to be true or known to be false or considered an open problem.

9.2 Famous formulas II. Consider the formula about integers with $+$ and $\cdot$ and integer constants given as

$$\exists x\, \forall y > x\, \exists v > 1 \exists w > 1\, [y - 1 = v \cdot w \vee y + 1 = v \cdot w].$$

Say in words what this formula says. Also determine by consulting Mathematics books or Wikipedia or otherwise, whether the formula is known to be true or known to be false or considered an open problem.

9.3 Famous formulas III. Consider the formula about integers with + and $\cdot$ and integer constants given as

$$\exists z \, \forall x \, \exists y > x \, \forall v > 1 \, \forall w > 1 \, [v \cdot w \neq y \wedge v \cdot w \neq y + z].$$

Say in words what this formula says. Also determine by consulting Mathematics books or Wikipedia or otherwise, whether the formula is known to be true or known to be false or considered an open problem.

9.4 Famous formulas IV. Consider the formula about integers with + and $\cdot$ and integer constants given as

$$\forall z > 1 \, \exists x > 1 \, \exists y > 1 \, \forall v > 1 \, \forall w > 1 \, [2z = x + y \wedge v \cdot w \neq x \wedge v \cdot w \neq y].$$

Say in words what this formula says. Also determine by consulting Mathematics books or Wikipedia or otherwise, whether the formula is known to be true or known to be false or considered an open problem.

9.5 Famous formulas V. Consider the formula about integers with + and $\cdot$ and integer constants given as

$$\forall x \, \exists y > x \, \forall v > 1 \, \forall w > 1 [v \cdot w \neq y].$$

Say in words what this formula says. Also determine by consulting Mathematics books or Wikipedia or otherwise, whether the formula is known to be true or known to be false or considered an open problem.

9.6 Famous formulas VI. Consider the formula about integers with + and $\cdot$ and integer constants given as

$$\forall x \, \forall y \, [(x + y) \cdot (x - y) = x \cdot x - y \cdot y].$$

Say in words what this formula says. Also determine by consulting Mathematics books or Wikipedia or otherwise, whether the formula is known to be true or known to be false or considered an open problem.

9.7 Derivations I. State the axioms of rings and explain how to derive the following formula from these axioms:

$$(x + y) \cdot (x + y) = x \cdot x + x \cdot y + y \cdot x + y \cdot y.$$

9.8 Derivations II. Consider the following axioms of groups:

$$\forall x \, \forall y \, \forall z \, [(x + y) + z = x + (y + z)]; \; \forall x \, [x + 0 = x \wedge 0 + x = x]; \; \forall x \, \exists y \, [x + y = 0];$$

$$\forall x \, \forall y \, \forall z \, [x + y = x + z \rightarrow y = z]; \; \forall x \, \forall y \, \forall z \, [y + x = z + x \rightarrow y = z].$$

Explain how to derive in first-order logic the following formula from these axioms:

$$\forall x \, \exists y \, [x + y = 0 \wedge y + x = 0].$$

Here 0 is the neutral element of the group with group operation $+$. The group is not known to be Abelian, so commutativity is not part of the axioms.

9.9 Axioms for Structures I. Provide a finite list of axioms with operations $+, \cdot$ and constants $0, 1$ so that all structure satisfying these axioms are infinite. The axioms should be consistent, so also provide some infinite model of it.

9.10 Axioms for Structures II. Provide some axioms for some relation R such that every structure satisfying these axioms is infinite and that every two countable structures satisfying these axioms are isomorphic.

9.11 Axioms for Structures III. Provide a list of axioms for a structure with constants a, b, c such that every model of these axioms has exactly three elements in its domain. Furthermore, for this structure, provide a formula α with two free variables x, y such that $\exists x\, \exists y\, [\alpha]$ is valid while substituting x by a and y by b provides a formula β which is false in the given model.

9.12 Axioms for Structures IV. Provide axioms which enforce that every finite structure satisfying these axioms has an even number of elements. Here the structure can use a unary function f, that is, f has one input variable. Is this set of axioms also satisfied by infinitely structures?

10.1 Proofs in First-Order Logic I. Recall the following rules for proving statements in first-order logic.

1. General Rules:
$$\frac{\alpha \in X}{X \vdash \alpha}, \frac{X \vdash \alpha}{X \cup Y \vdash \alpha};$$
2. Rules for And:
$$\frac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta}, \frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta};$$
3. Rules for Not:
$$\frac{X \vdash \alpha, \neg\alpha}{X \vdash \beta}, \frac{X, \alpha \vdash \beta | X, \neg\alpha \vdash \beta}{X \vdash \beta};$$
4. Rules for Quantifiers, where all variables in term t do not occur bounded in α at positions where x occurs free and where y does not occur free in $X \cup \{\alpha\}$:
$$\frac{X \vdash \forall x\, [\alpha]}{X \vdash \alpha\frac{t}{x}}, \frac{X \vdash \alpha\frac{y}{x}}{X \vdash \forall x\, [\alpha]}, \frac{X \vdash \alpha\frac{t}{x}}{X \vdash \exists x\, [\alpha]}, \frac{X \vdash \exists x\, [\alpha]}{X \vdash \neg\forall x[\neg\alpha]};$$
5. Rules for Equality, where all variables in terms s, u are not used for quantifiers in α:
$$\frac{}{X \vdash t = t}, \frac{X \vdash s = u, \alpha\frac{s}{x}}{X \vdash \alpha\frac{u}{x}}.$$

Use these rules to prove the following:
$$\forall y\, [x = y] \vdash \forall v\, \forall w\, [v = w].$$

10.2 Proofs in First-Order Logic II. Use the rules of Homework 10.1 to prove the following without any assumptions on X:

$$X \vdash \exists x \, \exists y \, [x = y].$$

10.3 Proofs in First-Order Logic III. Use the rules of Homework 10.1 to prove the following:

$$\forall x \, \forall y \, [f(x, y) = 0] \vdash f(2, 5) = 0.$$

10.4 Proofs in First-Order Logic IV. Use the rules of Homework 10.1 to prove the following where x, y do not occur free in X and 0 is a constant:

$$\text{If } X \vdash x = y \text{ then } X \vdash \forall z \, [z = 0].$$

10.5 Proofs in First-Order Logic V. Use the rules of Homework 10.1 to prove the following where x does not occur free in X and y does not occur in α:

$$\text{If } X \vdash \alpha \text{ then } X \vdash \alpha \frac{y}{x}.$$

10.6 Proofs in First-Order Logic VI. Use the rules of Homework 10.1 to prove the following where x, y do not occur free in X:

$$\text{If } X \vdash \alpha \text{ then } X \vdash \forall x \, [\alpha] \wedge \forall y \, [\alpha].$$

10.7 Deciding Logical Implication I. Does the following hold where α is an arbitrary formula:

$$\forall x \, [\alpha], \forall y \, [\alpha] \vdash \forall x \, \forall y \, [\alpha].$$

Either provide a proof using the rules of 10.1 or provide a model where the left side of $\vdash$ is satisfied but not the right side.

10.8 Deciding Logical Implication II. Does the following hold where s, t, u are arbitrary terms and x a variable:

$$s = t, t = u \vdash \forall x \, [s = u].$$

Either provide a proof using the rules of 10.1 or provide a model where the left side of $\vdash$ is satisfied but not the right side.

10.9 Deciding Logical Implication III. Does the following hold where a, b are constant names and x, y, z are variable names:

$$\forall x \, [x = a \vee x = b], y \neq z \vdash a \neq b.$$

Either provide a proof using the rules of 10.1 or provide a model where the left side of $\vdash$ is satisfied but not the right side.

10.10 Deciding Logical Implication IV. Assume that $(A, +)$ satisfies the set X of the following three axioms:

1. $\forall x \, \forall y \, \forall z \, [(x + y) + z = x + (y + z)]$,

2. $\forall x \, \forall y \, [x + y = x \vee x + y = y]$,

3. $\exists x \, \exists y \, [x \neq y]$.

Let

$$\alpha = \forall x \, \forall y \, [x + y = x] \vee \forall x \, \forall y \, [x + y = y]$$

and decide whether $X \vdash \alpha$ holds. Either provide a proof using the rules of 10.1 for $X \vdash \alpha$ or provide a model satisfying the axioms in X but not α.

11.1 Natural numbers and Successor. Consider the structure $(\mathbb{N}, 0, Succ)$ where $Succ(x) = x + 1$ in the standard model. Provide any nonstandard model of this structure with an explicit coding.

11.2 Dense linear order I. Consider the structure $(\mathbb{Q}, <)$ of a dense linear order without endpoints. Provide axioms for this structure such that all countable elements are isomorphic. Also provide an example of a uncountable non-standard model for this structure (it can be famous). Does this satisfy any first-order formulas not already satisfied by the standard model?

11.3 Dense linear order II. For the structure from 11.2, are there any models of it which do not contain a dense countable subset? If so, explain how to construct one, if not, say why they do not exist.

11.4 Constant structures I. Assume that there are constants $c_1, c_2, \ldots$ and the set X of all axioms saysing $c_i \neq c_j$ with $i \neq j$. Provide a nonstandard model for this structure which is a subset of all further nonstandard models. Explain why this property holds. Note that a nonstandard model contains some element which is not in the standard model given by the structure whose domain consists of all constants.

11.5 Constant structures II. For the model of 11.4, consider the cardinal $\aleph_1$ which is the first uncountable cardinal. By the Theorem of Löwenheim and Skolem, the structure from 11.4 and its axiom set X have a model of size $\aleph_1$. How many elements has this model which are equal to a constant and how many elements has this model different from all constants.

11.6 Injective functions I. Assume that a structure has exactly three elements x, y, z with $f(x) = y, f(y) = z, f(z) = x$ and all other elements u satisfy $f(u) = u$. Provide a set of axioms X for this structure, the formulas can use f as a mathematical symbol and $=$. Furthermore, how many models of sizes $\aleph_0$ and $\aleph_1$ does X have? Does X have finite models? If yes, of what sizes?

11.7 Injective functions II. Given X and f, introduce a further function g and one additional axiom: $\forall x \, [f(x) = g(g(x))]$. Show that there are various choices of g so that the several corresponding structures (A, f, g) coincide, when one removes g from the logical language.

11.8 Injective functions III. Assume that one has as axioms exactly one formula α: $\forall x \, [f(f(x)) = x \vee f(f(f(x))) = x]$. How many nonisomorphic structures exist for the cardinalities $1, 2, 3, 4$?

11.9 Injective functions IV. Assume that one has as axioms exactly one formula α: $\forall x \, [f(f(x)) = x \vee f(f(f(x))) = x]$. How many nonisomorphic structures exist for the cardinalities $5, 6, 7$?

11.10 Addition Machines I. A program is called an addition machine if it can do the following operations: Read a register from the input, write a register onto the output, Assign to a register the sum of various other registers and constants (with perhaps repetitions, the constants have to be integers), Assign to a register the difference of two terms which are either registers or constants or sums of registers and constants, Do unconditional or conditional branchings (gotos) where a conditional branching compares either two registers or one register and one constant with one of $<, =, >, \leq, \geq, \neq$. The program can have labels (line numbers) as places to jump to after a conditional branching or unconditional jump. One can also allow at if-then-else commands that instead of a Goto there is some sequence of commands between a begin and end to be done if the condition is satisfied (after the then) or not satisfied (after the else). The command Halt tells the machine to stop after writing the output. What does the following function compute?

1. Read x; If $x < 1$ then begin $x = -x$ end; $x = x + x + x + 1$; $z = 0$; $y = 1$;
2. If $y < x$ then begin $y = y + y + y$; goto 2 end;
3. If $x = y$ then Goto 7;
4. If $x \geq y$ then begin $x = x - y$, $z = z + 1$ end;
5. If $x \geq y$ then begin $x = x - y$, $z = z + 1$ end;
6. $x = x + x + x$; Goto 3;
7. Write z; Halt.

Provide the outputs for inputs 7 (ternary 21) and 28 (ternary 1001).

11.11 Addition Machines II. Provide an addition machine program which outputs 1 for an odd power of 2 and 2 for an odd power of 3 and 0 on all other inputs.

11.12 Addition Machines III. Provide an addition machine program which translates a binary number of the form $a_0 a_1 \ldots a_n$ into a binary number of the form $a_0 0 a_1 0 \ldots a_n$. For example, 7 (binary 111) is mapped to 21 (binary 10101).

11.13 Addition Machines IV. Provide an addition machine program which computes the number of binary digits of a positive input number to write down this number. For 0 and below, the output is 0 as a default. So input 7 gives output 3 and input 128 gives output 8.

11.14 Addition Machines V. Consider the following program.

1. Read x; Read y;
2. If $x < 0$ then $x = -1 - x - x$ else $x = x + x$;
3. If $y < 0$ then $y = -1 - y - y$ else $y = y + y$;
4. $v = 0$; $w = 0$; $u = x + y$;
5. If $v < u$ then begin $v = v + 1$; $w = w + v$; Goto 5 end;

6. $w = w + y$; Write w; Halt.

Prove that this program computes a one-one function from pairs of integers to natural numbers.

11.15 Addition Machines VI. Assume that the program in 11.4 computed from input x, y an output z. Provide an addition machine program which on input $z \geq 0$ finds out what x, y are and outputs x, y.

12.1 Recursion I. Assume that f, g are functions computed by register machines such that f has two inputs and g has four inputs. Now define $h(0, x, y) = f(x, y)$ and $h(z + 1, x, y) = g(z, h(z, x, y), x, y)$. Prove that also h can be computed by a register machine program. Use this example to explain that all primitive recursive functions can be computed by a register machine programs.

12.2 Recursion II. Assume that $f_e(x)$ is the e-th primitive recursive function with one input e where the program for f_e is generated by some computer from input e. Now consider the function $g(e) = 1 + f_e(e)$. Prove that g is not primitive recursive but can still be computed by some algorithm.

12.3 Unbounded Search I. Assume that some addition machine computes $f(x, y)$ on all inputs where this is defined. Now produce an addition machine program using f as a subprogram that computes a function $g(x)$ as follows: $g(x)$ takes the first value $z \geq 0$ found such that for all $y \in \{0, 1, \ldots, z - 1\}$ it holds that $f(x, y)$ is defined and different from 0 and $f(x, z)$ is defined and equal 0; if the search of the program for g never finds this z or if it runs onto a z where $f(x, z)$ does not halt then $g(z)$ is undefined, that is, its program runs forever.

12.4 Unbounded Search II. Given $f(x, y)$ as in 12.3 as a subprogram, write a function g which does the same search as in 12.3, but instead of z outputs $h(x, z)$ for some further subfunction h; if either the search described in 12.3 does not terminate with a z of if the program for $h(x, z)$ runs forever then $g(x)$ is undefined.

12.5 Enumerations of total functions I. Assume that $f(e, x, y)$ is a program which halts on all inputs. Construct a program $g(x, y)$ so that there is no e with $g(x, y) = f(e, x, y)$ for all x, y and such that $g(x, y)$ is defined for all inputs.

12.6 Enumerations of total functions II. Show that if there is a function $h(e, x, y)$ which decides where the e-th partial-recursive function with inputs x, y halts, then there is also a total function $f(e, x, y)$ such that $f(e, x, y)$ is the output of the e-th partial-recursive function with inputs x, y whenever this function is defined. Use the result of Homework 12.5 to show that there is a contradiction and that therefore the halting problem is undecidable. Besides 12.5, this homework can also use that there is a partial-recursive function $\psi(e, x, y)$ such that $\psi(e, x, y)$ takes the value of the e-th partial recursive function with x, y whenever this function is defined and provides some output.

12.7 Derivation of Formulas I. Assume that T is a consistent theory such that there is a partial recursive function f such that $f(x) = 1$ iff the x-th formula α satisfies $T \models \alpha$ and $f(x) = 0$ iff the x-th formula α satisfies $T \models \neg\alpha$; if neither $T \models \alpha$ nor $T \models \neg\alpha$ then $f(x)$ is undefined. Show that α is axiomatisable if and only if such an f exists.

12.8 Derivation of Formulas II. Show that T is decidable if the f from before exists and has a decidable domain. Furthermore, provide an example of a theory T where the f just mentioned has a decidable domain but where T is not complete.

12.9 Robinson's Arithmetic I. Robinson's Arithmetic Q has these axioms:

1. $\forall x\,[S(x) \neq 0]$;
2. $\forall x\,\forall y\,[S(x) = S(y) \rightarrow x = y]$;
3. $\forall x \neq 0\,\exists y\,[S(y) = x]$;
4. $\forall x\,[x + 0 = x]$;
5. $\forall x\,\forall y\,[x + S(y) = S(x + y)]$;
6. $\forall x\,[x \cdot 0 = 0]$;
7. $\forall x\,\forall y\,[x \cdot S(y) = (x \cdot y) + x]$.

Check whether the following structure satisfies the axioms of Q where the structure is defined by the following rules and domain: Domain $A = \mathbb{N} \cup \{\infty_1, \infty_2\}$, $S(x) = x + 1$ for $x \in \mathbb{N}$, $S(\infty_1) = \infty_1$, $S(\infty_2) = \infty_2$, $\forall x, y\,[x + y = y + x]$, $x + \infty_2 = \infty_2$, for $x \neq \infty_2$, $x + \infty_1 = \infty_1$, $\infty_i \cdot \infty_j = \infty_{\max\{i,j\}}$, $\infty_i \cdot S(x) = \infty_i$, $x \cdot \infty_i = \infty_i$, $\infty_i \cdot 0 = 0$ for $i, j \in \{1, 2\}$ and $x \in \mathbb{N}$. Provide a formula which is satisfied by this structure, but not by the substructure with domain $B = \mathbb{N} \cup \{\infty_1\}$ (that substructure satisfies Q by Rautenberg's book).

12.10 Robinson's Arithmetic II. Is there a structure with the same domain A as 12.9 which satisfies Q and where the restriction to $\mathbb{N}$ is identical with the usual operations $S, +, \cdot$ on the natural numbers and where this model is not isomorphic to the one in 12.9, also not by swapping the names of ∞_1 and ∞_2?

13.1 Categoricity I. Recall that a theory is κ-categorical if the theory, up to isomorphism, has exactly one model of size κ. Assume that the logical language has one function symbol f and a theory T is axiomatised by the following single axiom:

$$\forall x\,[f(x) \neq x \wedge f(f(x)) = x].$$

For which cardinals κ is the theory T κ-categorical?

13.2 Categoricity II. Assume that the logical language has one function symbol f and a theory T is axiomatised by the following single axiom:

$$\forall x\,[f(x) \neq x \wedge (f(f(x)) = x \vee f(f(f(x))) = x)].$$

For which cardinals κ is the theory T κ-categorical?

13.3 Categoricity III. Assume that the logical language has one function symbol f and a theory T is axiomatised by the following single axiom:

$$\forall x\,\forall y\,[f(f(x)) = x \wedge (y \neq x \rightarrow f(y) = x)].$$

For which cardinals κ is the theory T κ-categorical?

13.4 Categoricity IV. Assume that the logical language has one function symbol P and a theory T is axiomatised by the following single axiom:

$$\exists x \, \exists y \, \forall z \, [P(x) \wedge P(y) \wedge (z \neq x \wedge z \neq y \rightarrow \neg P(z))].$$

For which cardinals κ is the theory T κ-categorical?

13.5 Number of Models I. Assume the logical language contains constants $c_0, c_1, \ldots$ and one order predicate $<$. Let T contain all axioms of a dense linear order without endpoints and furthermore, for each index k of a constant, the following additional axioms: $c_k < c_{k+1} \rightarrow c_{k+1} < c_{k+2}$, $\neg(c_k < c_{k+1}) \rightarrow c_{k+1} = c_{k+2}$, $c_k = c_{k+1} \rightarrow c_1 = c_2$. How many countable models does this structure have?

13.6 Number of Models II. Assume the logical language contains constants $c_0, c_1, \ldots$ and one order predicate $<$. Let T contain all axioms of a dense linear order without endpoints and furthermore, for each index k of a constant, the following additional axioms: $\neg(c_{k+1} < c_k)$, $c_k = c_{k+1} \rightarrow c_{k+1} = c_{k+2}$, $c_k = c_{k+1} \rightarrow c_2 = c_3$. How many countable models does this structure have?

13.7 Number of Models III. Assume the logical language contains constants $c_0, c_1, \ldots$ and one order predicate $<$. Let T contain all axioms of a dense linear order without endpoints and furthermore, for each index k of a constant, the following additional axioms: $c_k < c_{k+1} \rightarrow c_{k+1} < c_{k+2}$, $\neg(c_{k+1} < c_k)$, $c_k = c_{k+1} \rightarrow c_{k+1} = c_{k+2}$. How many countable models does this structure have?

13.8 Number of Models IV. Let $\circ$ be a semigroup operation on a domain D, that is, D is associative. There are no further requirements. How many models with $|D| = 2$ exist (up to isomorphism)?

13.9 Number of Models V. Let $\circ$ be a monoid operation on a domain $D = \{0, 1\}$, that is, D is associative and has neutral element 0. There are no further requirements. The constants $0, 1$ are part of the logical language. How many models with domain D exist?

13.10 Number of Models VI. Let $\circ$ be a monoid operation on a domain $D = \{0, 1, 2\}$, that is, D is associative and has a neutral element, here it should be the 0. The constants $0, 1, 2$ are part of the logical language. How many models with domain D exist?

13.11 Repetition: First-Order Description I. Provide axioms which describe the requirements for the models in Homework 13.9 in First-Order Logic formally.

13.12 Repetition: First-Order Description II. Provide axioms which describe the requirements for the models in Homework 13.10 in First-Order Logic formally.