Members of thin Π^0_1 classes and their Turing degrees

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Π^0_1 classes

Definition:

A Π^0_1 class is a set $P \subseteq 2^\omega$ for which there is a (primitive) recursive tree T with $[T] = P$.

- As primitive recursive trees can be effectively enumerated, we have an effective enumeration of Π^0_1 classes.

Examples:

- Consider $A = \{e : \varphi_e(0) \downarrow= 0\}$ and $A = \{e : \varphi_e(0) \downarrow= 1\}$.
  - A and B are disjoint r.e. sets, and cannot be recursively separated.
  - The class of sets separating A and B

    $$S(A, B) = \{C : A \subseteq C \ \& \ B \cap C = \emptyset\}$$

    is called the separating class of A and B.
  - $S(A, B)$ is a Π^0_1 class and is perfect (hence uncountable).
- The class of complete consistent extension of Peano Arithmetic is a Π^0_1 class.
- Zariski topology over recursive rings, where for any r.e. ideal I, the collection of prime ideals containing I forms a Π^0_1 class.

Thin Π^0_1 classes

Definition: Thin Classes
A Π^0_1 class P is thin if every subclass of P is relatively clopen, i.e., if Q is a subclass of P, then $Q = P \cap U$ for some clopen set $U \subseteq 2^\omega$.

We know that in all Π^0_1 classes, isolated paths are computable.

Conversely, if a thin Π^0_1 class P contains a computable element X, then $\{X\}$ is a subclass of P, and hence by the thinness of P, X is isolated.

FACT:
A thin Π^0_1 class P has no computable members if and only if P is perfect.

So every countable thin Π^0_1 class has a computable member.

Martin-Pour El theories

The notion of thinness comes from the work of Martin and Pour-El in 1970. Let S be a consistent r.e. theory in the propositional language with

Martin and Pour-El, 1970

Let S be a consistent r.e. theory.

(1) S has few r.e. extensions if each r.e. extension T of S is a principal extension, i.e., T is generated by S together with a single propositional formula.

(2) S is essentially undecidable if S has no decidable complete consistent extensions.

FACTS:

For a consistent r.e. theory S,

- S has few r.e. extensions if and only if the corresponding Π^0_1 class is thin.
- S is essentially undecidable if and only if the corresponding Π^0_1 class has no computable members.

Turing degrees of members of thin Π^0_1 classes

Theorem (CDJS, 1993):

If X is in a thin Π^0_1 class P, then $X' \leq_T X \oplus \emptyset''$.

Proof: Let $P = [T]$ is a thin class, where T is a recursive tree, and $A \in P$.

For a given e, we consider whether $e \in A'$ or not, i.e., whether $\Phi^A_e(e)\downarrow$ or not.

- If $\Phi^A_e(e)\downarrow$, we can recursive in A to find an initial segment σ of A with $\{e\}^\sigma(e)\downarrow$.
- If NOT, what shall we do?

Consider $Q_e = \{C : \Phi_e^C(e)\uparrow\}$, a Π_1^0 class

- $P \cap Q_e$ is a subclass of P, and as P is thin, $P \cap Q_e = P \cap U_e$ for some clopen set U_e.
- As we are assuming that A is in $P \cap Q_e$, A is in $P \cap U_e$, and hence A has an initial segment σ with all infinite extensions in U_e.

  Thus, if $B \in P$ extends σ, then $B \in P \cap U_e = P \cap Q_e$, and $\Phi_e^B(e)\uparrow$.
- Define a binary relation $R(e,\sigma)$ as

$$R(e,\sigma) \iff (\forall \tau \supseteq \sigma)[\tau \in T \ \& \ \{e\}^\tau(e)\downarrow \to \ \tau \notin Ext(T)].$$

  R is a Π_2 relation and is recursive in $\emptyset''$.

We do as following:

Find the least number n such the following is true for $\sigma = A \upharpoonright n$:

(a) $\{e\}^\sigma(e)\downarrow \quad \longrightarrow e \in A'$

(b) $R(e,\sigma) \quad \longrightarrow e \notin A'$

Exact one of these will appear.

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If A computes $\emptyset''$, then A cannot be a member of any thin Π_1^0 class.

Spector's construction

- In Spector's construction of minimal degrees below $\mathbf{0}''$, forcing notions are recursive perfect trees, T_e, $e \in \omega$, pruned according to the black-white rule.

  That is, to see whether we can find a string $\sigma \in T_e$ such that there is no e-splitting above σ in T_e, or not.

- If we use only 'half' of each T_e, i.e., keep the even part, and exclude the odd part, the construction still works.

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A great observation.

Construct one thin-free degree below $\mathbf{0}''$

Definition:

A Turing degree is thin-free, if no members in this degree is a member of thin Π^0_1 classes.

Note that all degrees above $\mathbf{0}''$ are thin-free.

We will construct a set A of thin-free degree below $\mathbf{0}''$, we shall ensure for any e such that if Φ^A_e is total and Turing equivalent to A, then one of the following is guaranteed:

(1) $\Phi^A_e \notin [P_e]$, or

(2) $[P_e]$ is not thin.

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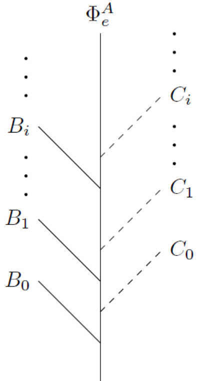
Suppose that A is constructed on a given recursive perfect tree T.

- To meet (1), we try to find some string τ on T such that Φ_e^τ is not extendible on P_e,
  - if such a τ exists, we force A to extend τ, which guarantees that $\Phi_e^A \notin [P_e]$, if Φ_e^A is total.

If NOT, we will then try to

- find a Π^0_1 subclass of $[P_e]$ which is not the intersection of $[P_e]$ with any clopen set U.

  We will construct a recursive subtree S_e of P_e, such that Φ_e^A lies on S_e, and for any length n, there exists some $B \in [S_e]$ and $C \in [P_e] \backslash [S_e]$ such that
  $$B \upharpoonright n = C \upharpoonright n = \Phi_e^A \upharpoonright n.$$
  This implies that $\Phi_e^A \in [P_e]$, and $[S_e]$ witnesses that P_e is not thin.



Action under this case:

- Target: Force A on a total recursive subtree T_e of T, such that for any $\alpha \in T_e$, $\Phi_e^{T_e(\alpha\hat{}0)}$ and $\Phi_e^{T_e(\alpha\hat{}1)}$ are incompatible in P_e and there is a path on P_e extending $\Phi_e^{T_e(\alpha)}$, of course.

  We are assuming that (1) fails, so both $\Phi_e^{T_e(\alpha\hat{}0)}$, $\Phi_e^{T_e(\alpha\hat{}1)}$ are extendible on P_e and thus there is at least one infinite path in P_e extending it.

Consider the e-splitting subtree of T, $SP(T, e)$, if exists, and take the even part.

- White Side: $SP(T, e)$ exists.

  In this case, Φ_e^A is total, then Φ_e^A is on $[P_e]$, and $E(SP(T, e))$, the even subtree of $SP(T, e)$, is a total recursive subtree of T, and $\Phi_e^{E(SP(T,e))}$ is a total recursive subtree of P_e, witnessing that $[P_e]$ is not thin.

- **Black Side**: $SP(T, e)$ does not exist.

  In this case, there is a string $T(\alpha)$ such that above $T(\alpha)$, no string e-splits, and hence, if A is on the full subtree of T above α, $Full(T, \alpha)$, then Φ_e^A is recursive, making A and Φ_e^A not Turing equivalent, if we can make A nonrecursive. **We Can**, as recursive sets are all in thin Π_1 classes..

Oracle Construction:

We can now run a forcing argument to construct A with wanted property.

- $\mathbf{0}''$ is used as oracle to make decision at every stage.

Yuan Bowen improved this in his thesis:

Theorem:
There exists a hyperimmune-free minimal degree below $\mathbf{0}''$ which is also thin-free.

Note that such degrees are not below $\mathbf{0}'$.

Working below $\mathbf{0}'$

- CDJS proved that $\mathbf{0}'$ contains a Π_1 set A which is in a thin Π_1 class P.
- CDJS proved the density of degrees containing sets (not necessarily r.e.) in thin Π_1 classes in r.e. degrees.

  DWY strengthened this in 2018, showing that sets above can be r.e.
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- Yuan Bowen proved in his thesis that all 1-generic degrees below $\mathbf{0}'$ contain members of thin Π_1 classes.
- There are degrees below $\mathbf{0}'$ thin-free and then can be r.e., or minimal, by CDJS.

The construction of a minimal thin-free degree was given by CDJS, modified from Sacks forcing, where partial recursive trees are used.

An r.e. thin-free degree

Construct an r.e. A satisfying the following requirements:

$\mathcal{R}_e$: if $\Phi_e(A)$ and $\Psi_e(\Phi_e(A))$ are both total, then either

- $A \neq \Psi_e(\Phi_e(A))$; or
- $\Phi_e(A)$ is not in $[P_e]$; or
- $[P_e]$ is not thin.

In this construction, we cannot use the e-splitting tree as a help to construct a subclass witness that $[P_e]$ is not thin.

We thus need to construct such a subclass, actually, a subtree, by infinitely many substrategies, each of which tries to find an infinite path in $[P_e]$, and

- any substrategy fails to secure an infinite path, an enumeration of a certain number into A, showing that either $\underline{A \neq \Psi_e(\Phi_e(A))}$ (diagonalization succeeds) or $\underline{\Phi_e(A) \text{ is not in } [P_e]}$, a global win for $\mathcal{R}_e$.

DWY proved in 2018 that such r.e. degrees are dense in the r.e. degrees.

Other topics

In his thesis, Yuan proved that any nonrecursive set below a 2-generic set is thin-free. In particular, 2-generic degrees are thin-free.

CDJS also consider minimal Π^0_1 classes and Cantor-Bendixson rank of sets, a topic originated from Cenzer, et al.'s work in 1986.

Our continuing work on this topic is in the direction of Ershov hierarchy, also 1-generic degrees not below $\mathbf{0}'$, *pb*-generic degrees, minimal degrees with full approximations.

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Thanks!

Thanks!

Take care and keep safe!