HANDOUT
Singapore Logic Seminar, 03-21-2021

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This talk is based on material published in [1, 2, 3, 4] and some unpublished material.

> *Do we need, or do we not need, unbounded search in order to convert one representation of an irrational number into another representation?*

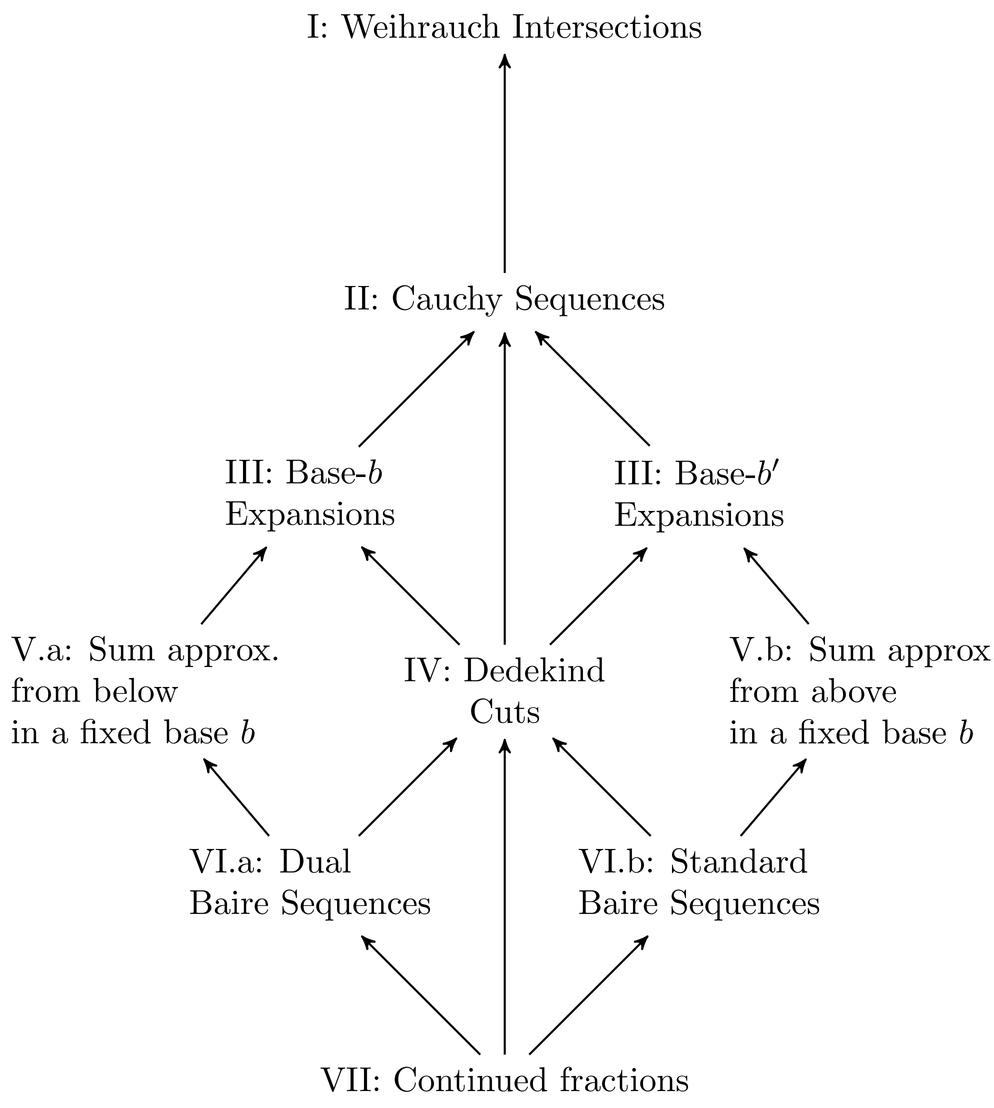
$\vdots$

> *Do we need, or do we not need, unbounded search in order to convert a Cauchy sequence for α into the Dedekind cut of α?*
>
> *Do we need, or do we not need, unbounded search in order to convert the Dedekind cut of α into a Cauchy sequence for α?*
>
> *. . . the base-2 expansion of α into the base-10 expansion of α?*
>
> *. . . the base-10 expansion of α into the base-2 expansion of α?*
>
> *. . . the continued fraction of α into the base-17 expansion of α?*
>
> *. . . the Dedekind cut of α into the the continued fraction of α?*

$\vdots$

A computation that does not apply unbounded search is called a *subrecursive* computation. Primitive recursive computations and (Kalmar) elementary computations are typical examples of subrecursive computations. A representation R_1 (of irrational numbers) is *subrecursive in* a representation R_2 if the R_1-representation of α can be subrecursively computed in the R_2-representation of α. Two representations R_1 and R_2 are *equivalent* when R_1 is subrecursive in R_2 and R_2 is subrecursive in R_1.

Overview of subrecursive reducibility among (equivalence classes of) representations:



I: Representations Equivalent to Weihrauch Intervals

Let α be an irrational number in the interval $(0, 1)$.

Weihrauch Intervals

A function $I_{\cdot} : \mathbb{N} \to \mathbb{Q} \times \mathbb{Q}$ is a *Weihrauch intersection* of α if

$$\{\ \alpha\ \} \ = \ \bigcap_n I_n^O$$

where I_n^O denotes the open interval given by the the pair $I(n)$.

Comments: This is a representation from Weihrauch's book [19].

Nested Weihrauch Intervals

A *nested Weihrauch intersection* $I : \mathbb{N} \to \mathbb{Q} \times \mathbb{Q}$ is a Weihrauch intersection such that I_{n+1}^O is a strict subinterval of I_n^O.

Complete Topological Names

Let $f : \mathbb{N} \to \mathbb{Q} \times \mathbb{Q}$ be such that (1) for any open interval I with rational endpoints and $\alpha \in I$ exits n such that $f(n) = (r_1, r_2) = I$ and (2) if $x \notin I$, we have $f(n) \neq I$ for all n. So $\{f(i)\}_{i \in \mathbb{N}}$ is a sequence whose elements are exactly the open intervals with rational endpoints that contains α, and we have $\{\alpha\} = \bigcap_i f(i)$. Then we say that f is a *complete topological name* for α.

Comments: This is a representation from Weihrauch's book [19].

II: Representations Equivalent to Cauchy Sequences

Let α be an irrational number in the interval $(0,1)$.

Cauchy Sequences

The function $C : \mathbb{N} \to \mathbb{Q}$ is a *Cauchy sequence* for α if

$$|\alpha - C(n)| \; < \; \frac{1}{2^n}.$$

Strictly Increasing Cauchy Sequences

The function $C : \mathbb{N} \to \mathbb{Q}$ is a *strictly increasing Cauchy sequence* for α if (i) C is a Cauchy sequence for α and (ii) $C(n) < C(n+1)$.

Base-b Cauchy Sequences

Let $b \in \mathbb{N} \setminus \{0,1\}$. The function $f : \mathbb{N} \to \mathbb{N}$ is a *base-b Cauchy sequence* for α if

$$C(n) := \frac{f(n)}{b^n}$$

is a Cauchy sequence for α.

Comments: Friedman & Ko [8] use base-2 Cauchy sequences. They call them "converging sequences of dyadic rational numbers". *Increasing base-b Cauchy sequences* will not be equivalent to base-b Cauchy sequences.

Fuzzy (Dedekind) Cuts

The function $D : \mathbb{N} \times (\mathbb{N} \setminus \{0\}) \to \{0,1\}$ is a *fuzzy Dedekind cut* for α if

$$D\Big(\; m, n \;\Big) = 0 \;\Rightarrow\; \alpha < \frac{m+1}{n} \qquad \text{and} \qquad D\Big(\; m, n \;\Big) = 1 \;\Rightarrow\; \frac{m-1}{n} < \alpha \; .$$

Signed Digit Expansions

The function $S : (\mathbb{N} \setminus \{0,1\}) \to \{-1,0,1\}$ is a *signed digit expansion* of α if $S(0) = 0$ and

$$\alpha = \sum_{i=1}^{\infty} \frac{S(i)}{2^i} \; .$$

Comments: This seems to be a pretty standard representation, see Berger et al. [7].

III: Equivalence Classes of Base-b Expansions

Let α be an irrational number in the interval $(0, 1)$.

The function $f : \mathbb{N} \rightarrow \{0, \ldots, b-1\}$ is the *base-b expansion* of α if $E(0) = 0$ and

$$\alpha \;=\; \sum_{i=1}^{\infty} \frac{f(i)}{b^i} \;.$$

We use E_b^α to denote the base-b expansion of α.

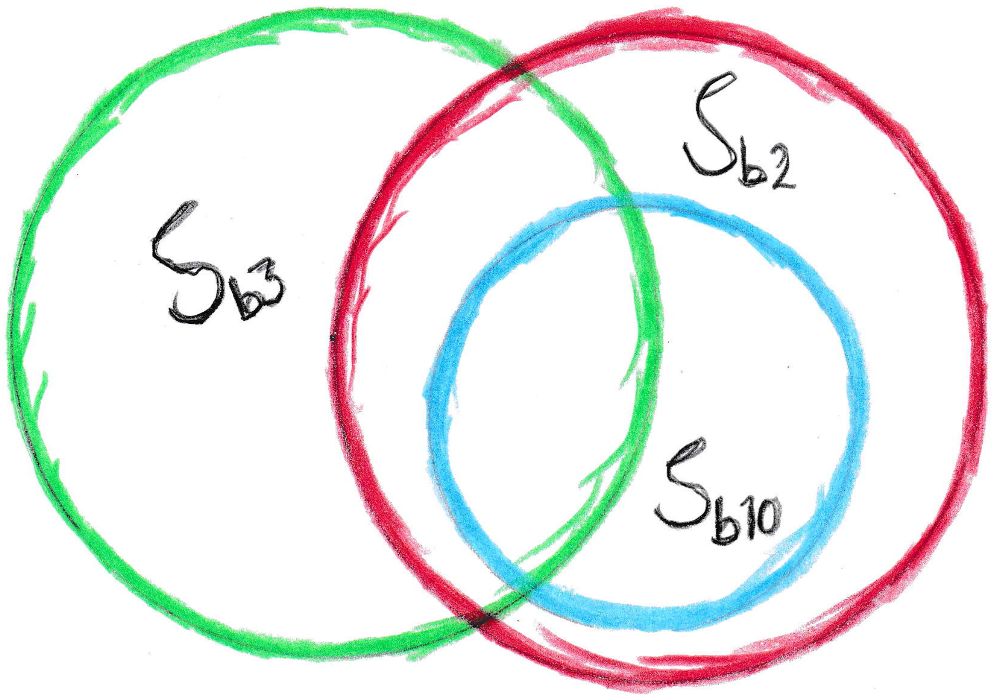
The next theorem implies that there is is a lot of degrees between between the degree of the Cauchy sequences and the degree of the Dedekind cuts.

Theorem 1 *The base-b expansion is subrecursive in the base-b' expansion if and only if every prime that divides b also divides b'.*

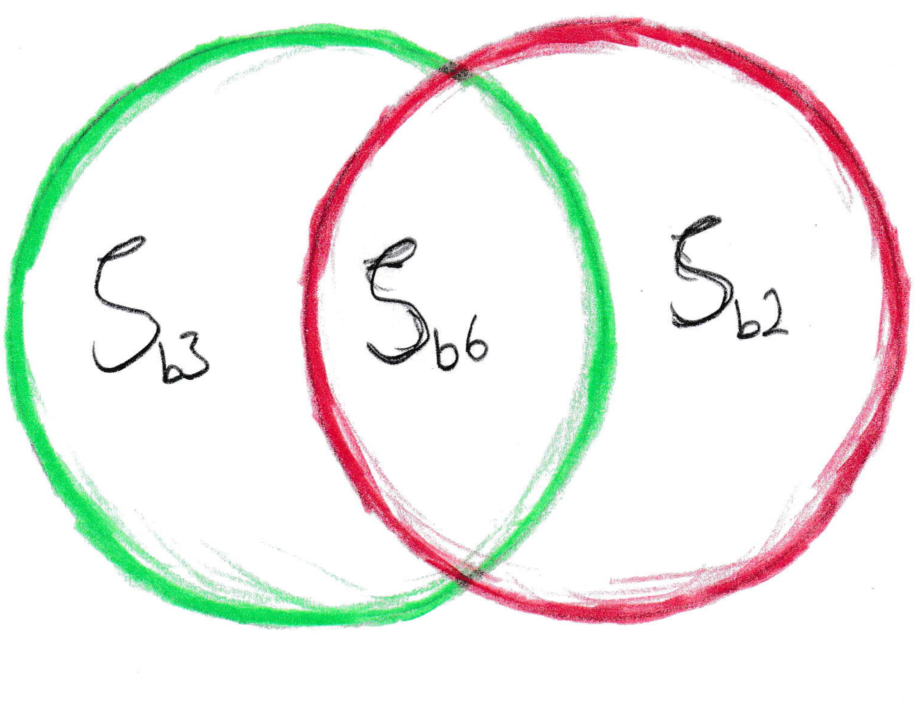
Comments: Every prime that divides b will also divides b' if and only if every rational number that has a finite base-b expansion also has a finite base-b' expansion.

We may also consider Venn diagrams. In the Venn diagrams AT THE NEXT PAGE we use the notation:

- $\mathcal{S}$ may be any subrecursive class closed under elementary operation
- $\mathcal{S}_{bn}$ is the class of irrational numbers that have a base-n expansion in $\mathcal{S}$.



Sb2
Sb3
Sb10



Sb3
Sb6
Sb2

IV: Representations Equivalent to Dedekind Cuts

Let α be an irrational number in the interval $(0, 1)$.

Dedekind Cuts

The function $D : \mathbb{Q} \to \{0, 1\}$ is the *Dedekind cut* of α if

$$D(q) = \begin{cases} 0 & \text{if } q < \alpha \\ 1 & \text{if } \alpha < q. \end{cases}$$

General Base Expansions

The function $E : (\mathbb{N} \setminus \{0, 1\}) \times \mathbb{N} \to \{0, \ldots, b-1\}$ is the *general base expansion* of α if $E(b, n) = E^{\alpha}_{b}(n)$ (the base-b expansion E^{α}_{b} is defined above above).

Beatty Sequences

The function $B : (\mathbb{N} \setminus \{0\}) \to \mathbb{N}$ is the *Beatty sequence* of α if

$$\frac{B(n)}{n} \; < \; \alpha \; < \; \frac{B(n)+1}{n} \; .$$

Comments: The name *Beatty sequence* has its origin in the publication [5]. Apparently, what is now known as Beatty sequences was used earlier by Bernard Bolzano [6], whence this representation could also be called *Bolzano measures.*

Hurwitz Characteristics

For any string $\tau \in \{L, R\}^*$, we define the *interval addressed by* τ inductively over the structure of τ: The empty sequence addresses the interval $(0/1, 1/1)$. Furthermore

$$\tau L \text{ addresses } \left(\frac{a}{b}, \frac{a+c}{b+d} \right) \text{ and } \tau R \text{ addresses } \left(\frac{a+c}{b+d}, \frac{c}{d} \right)$$

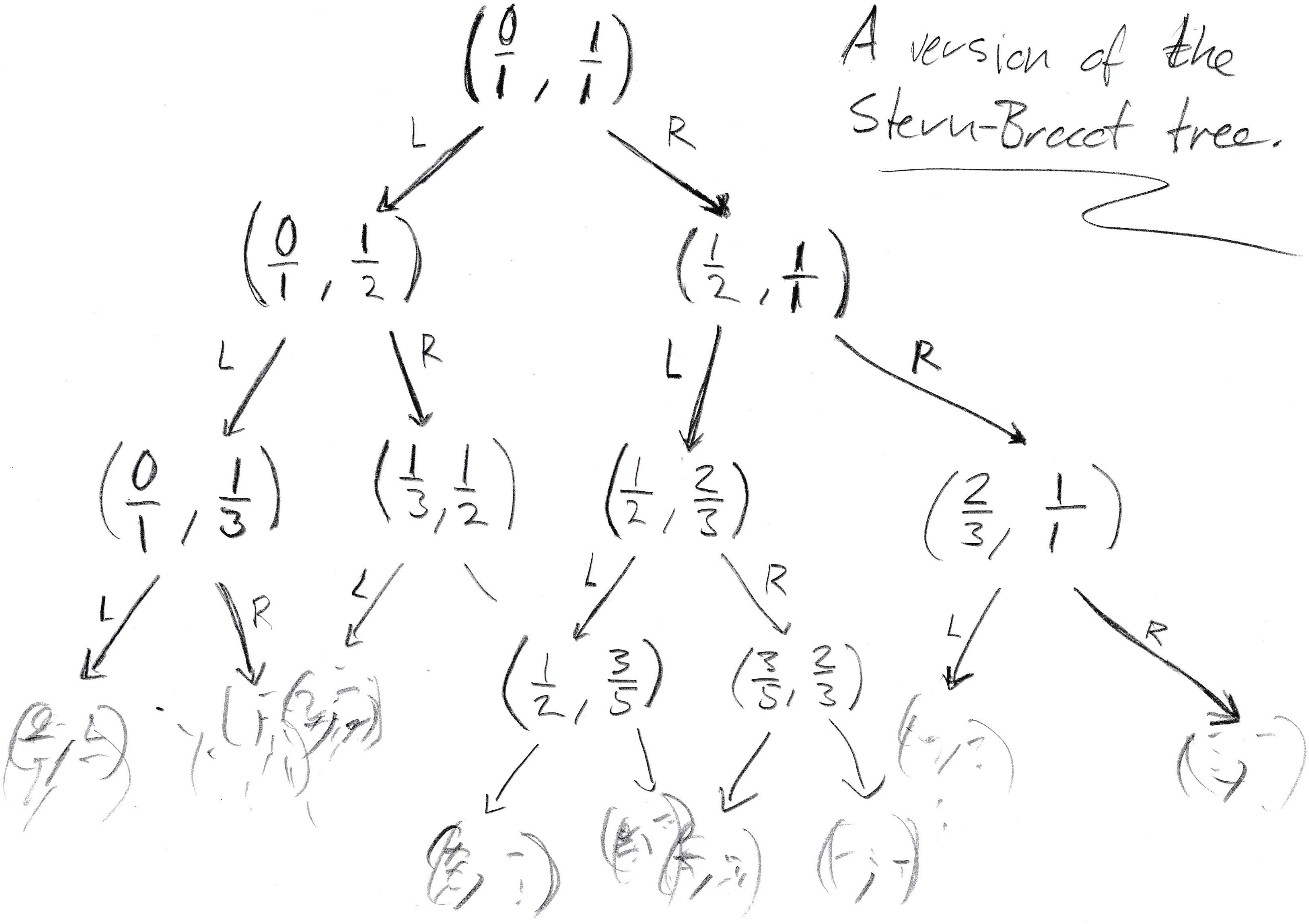
if τ addresses $(a/b, c/d)$.

The infinite sequence Σ over the alphabet $\{L, R\}$ addresses α if any finite prefix of Σ addresses an interval that contains α.

Let Σ^{α} address α. The function $H : (\mathbb{N} \setminus \{0\}) \to \{0, 1\}$ is the *Hurwitz characteristic* of α if

$$H(n) = \begin{cases} 0 & \text{if the } n\text{'th element of } \Sigma^{\alpha} \text{ is } L \\ 1 & \text{if the } n\text{'th element of } \Sigma^{\alpha} \text{ is } R. \end{cases}$$

Comments: Hurwitz characteristics were known in the 19th century, see Hurwitz [10]. For more on Hurwitz characteristics (as representation of irrationals) see Lehman [14] and Kristiansen & Simonsen [4]. A Hurwitz characteristic yields a branch in the Stern-Brocot tree, see the figure AT THE NEXT PAGE.


A version of the Stern-Brocot tree.
(0/1, 1/1)
L
R
(0/1, 1/2)
(1/2, 1/1)
(0/1, 1/3)
(1/3, 1/2)
(1/2, 2/3)
(2/3, 1/1)
(1/2, 3/5)
(3/5, 2/3)

V: Equivalence Classes of Sum Approximations

Let α be an irrational number in the interval $(0, 1)$.

Let $0.\mathtt{D}_1\mathtt{D}_2\ldots$ be the base-b expansion of α (and recall that $E_b^\alpha(i) = \mathtt{D}_i$). If $\mathtt{D}$ is a base-b digit, then $\overline{\mathtt{D}}$ denotes the *complement digit* of $\mathtt{D}$, that is, $\overline{\mathtt{D}} = (b-1) - \mathtt{D}$.

The *base-b sum approximation from below* of α is the function $\hat{A}_b^\alpha : \mathbb{N} \to \mathbb{Q}$ defined by $\hat{A}_b^\alpha(0) = 0$ and

$$\hat{A}_b^\alpha(n+1) \quad = \quad \frac{E_b^\alpha(m)}{b^m}$$

where m is the least m such that

$$\sum_{i=0}^{n} \hat{A}_b^\alpha(i) \quad < \quad 0.\mathtt{D}_1\ldots\mathtt{D}_m\ .$$

The *base-b sum approximation from above* of α is the function $\check{A}_b^\alpha : \mathbb{N} \to \mathbb{Q}$ defined by $\check{A}_b^\alpha(0) = 0$ and

$$\check{A}_b^\alpha(n+1) \quad = \quad \frac{\overline{E_b^\alpha(m)}}{b^m}$$

where m is the least m such that

$$1 \ - \ \sum_{i=0}^{n} \check{A}_b^\alpha(n) \quad > \quad 1 \ - \ 0.\overline{\mathtt{D}}_1\ldots\overline{\mathtt{D}}_m\ .$$

Observe that we have

$$\sum_{i=0}^{\infty} E_b^\alpha(n) \quad = \quad \sum_{n=0}^{\infty} \hat{A}_b^\alpha(n) \quad = \quad 1 \ - \ \sum_{n=0}^{\infty} \check{A}_b^\alpha(n)\ .$$

Example: Let the base-10 expansion of α start with the digits $0.3000604\ldots$. Then we have

$$\hat{A}_b^\alpha(1) = 3 \times 10^{-1} \qquad \hat{A}_b^\alpha(2) = 6 \times 10^{-5} \qquad \hat{A}_b^\alpha(3) = 4 \times 10^{-7}$$

and

$$\check{A}_b^\alpha(1) = 6 \times 10^{-1} \qquad \check{A}_b^\alpha(2) = 9 \times 10^{-2} \qquad \check{A}_b^\alpha(3) = 9 \times 10^{-3}\ .$$

Exercise: Assume that the base-b expansion of α contains very long sequences of zeros. Describe the base-b sum approximation from below of α. Describe the base-b sum approximation from above of α.

We say that a representation R_1 is *incomparable* to a representation R_2 if (i) R_1 is not subrecurisve in R_2 and (ii) R_2 is not subrecurisve in R_1. The next theorems show that there is a lot of degrees that are incomparable to the degree of the Dedekind cut.

Theorem 2 *Let b be an arbitrary base (so $b \geq 2$). Then*

- *the base-b sum approximation from below is incomparable to the base-b sum approximation from above*
- *the base-b sum approximation from below is incomparable to the Dedekind cut*
- *the base-b sum approximation from above is incomparable to the Dedekind cut.*

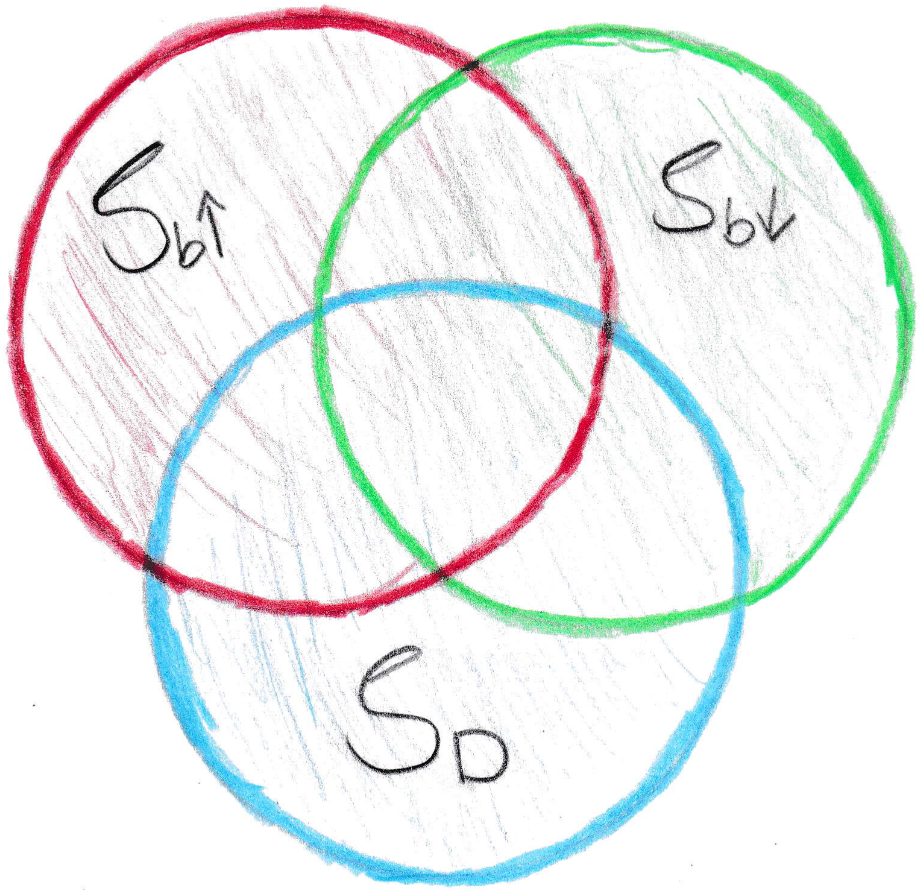
Theorem 3 *The base-b sum approximation from below of α is subrecursive in the base-b' sum approximation from below of α if and only if every prime that divides b also divides b'.*

Theorem 4 *The base-b sum approximation from above of α is subrecursive in the base-b' sum approximation from above of α if and only if every prime that divides b also divides b'.*

Comments: Observe the similarities between the two preceding theorems and Theorem 1.

The Venn diagram AT THE NEXT PAGE gives a little bit more informative than Theorem 2. In the diagram we use the notation:

- $\mathcal{S}$ may be any subrecursive class close under elementary operations
- $\mathcal{S}_D$ is the class of irrational numbers that have a Dedekind cut in $\mathcal{S}$
- $\mathcal{S}_{b\uparrow}$ is the class of irrational numbers that have base-b sum approximation from below in $\mathcal{S}$
- $\mathcal{S}_{b\downarrow}$ is the class of irrational numbers that have base-b sum approximation from above in $\mathcal{S}$


$S_{b\uparrow}$
$S_{b\downarrow}$
S_D

VI: Representations Equivalent to (Standard and Dual) Baire Sequences

Let α be an irrational number in the interval $(0,1)$.

Standard Baire Sequences

Let $f : \mathbb{N} \to \mathbb{N}$ be any function, and let $n \in \mathbb{N}$. We define the interval I_f^n by $I_f^0 = (0/1, 1/1)$ and

$$I_f^{n+1} \;=\; \left(\frac{a + f(n)c}{b + f(n)d} \,,\, \frac{a + f(n)c + c}{b + f(n)d + d} \right)$$

if $I_f^n = (a/b, c/d)$. The function $B : \mathbb{N} \to \mathbb{N}$ is the *standard Baire representation* of α if we have $\alpha \in I_B^n$ for every n.

Dual Baire Sequences

We define the interval J_f^n by $J_f^0 = (0/1, 1/1)$ and

$$J_f^{n+1} \;=\; \left(\frac{a + f(n)a + c}{b + f(n)b + d} \,,\, \frac{f(n)a + c}{f(n)b + d} \right)$$

if $J_f^n = (a/b, c/d)$. The function $A : \mathbb{N} \to \mathbb{N}$ is the *dual Baire representation* of α if we have $\alpha \in J_A^n$ for every n.

General Sum Approximations from Below

The *general sum approximation from below* of α is the function

$$\hat{G}^\alpha : ((\mathbb{N} \setminus \{0,1\}) \times \mathbb{N}) \to \mathbb{Q}$$

given by

$$\hat{G}^\alpha(b,x) \;=\; \hat{A}_b^\alpha(x)$$

where $\hat{A}_b^\alpha$ is the base-b sum approximation from below of α (see definition above).

General Sum Approximations from Above

The *general sum approximation from above* of α is the function

$$\check{G}^\alpha : ((\mathbb{N} \setminus \{0,1\}) \times \mathbb{N}) \to \mathbb{Q}$$

given by

$$\check{G}^\alpha(b,x) \;=\; \check{A}_b^\alpha(x)$$

where $\check{A}_b^\alpha$ is the base-b sum approximation from above of α (see definition above).

Left Best Approximations

Let a and b be relatively prime natural numbers with $b > 0$. The fraction a/b is a *left best approximant* of α if we have $c/d \leq a/b < \alpha$ or $\alpha < c/d$ for any natural numbers c, d with $0 < d \leq b$. A *left best approximation* of α is a sequence of fractions $\{a_i/b_i\}_{i\in\mathbb{N}}$ such that

$$(0/1) = (a_0/b_0) < (a_1/b_1) < (a_2/b_2) < \dots$$

and each a_i/b_i is a left best approximant of α.

Right Best Approximations

Let a and b be relatively prime natural numbers with $b > 0$. The fraction a/b is a *right best approximant* of α if we have $\alpha < a/b \leq c/d$ or $c/d < \alpha$ for any natural numbers c, d with $0 < d \leq b$. A *right best approximation* of α is a sequence of fractions $\{a_i/b_i\}_{i\in\mathbb{N}}$ such that

$$(1/1) = (a_0/b_0) > (a_1/b_1) > (a_2/b_2) > \dots$$

and each a_i/b_i is a right best approximant of α.

Theorem 5 *The representations*

- *Dual Baire Sequences*
- *Left Best Approximations*
- *General Sum Approximations from Below*

are equivalent. The representations

- *Standard Baire Sequences*
- *Right Best Approximations*
- *General Sum Approximations from Above*

are equivalent. Moreover, any representation in the first equivalence class is incomparable to any representation in the second equivalence class.

VII: Representations Equivalent to Continued Fractions

Let α be an irrational number in the interval $(0,1)$.

Comments: Continued fractions are well known from the literature. The continued fraction $[\ 0\ ; a_1, a_2, \ldots\ ]$ of α is the unique sequence of positive integers such that

$$\alpha \ = \ 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_2 + \ldots}}}$$

Continued Fractions

The function $f : (\mathbb{N} \setminus \{0\}) \to (\mathbb{N} \setminus \{0\})$ is the *continued fraction* of $\alpha \in (0,1)$ if $\alpha = [\ 0\ ; f(1), f(2), f(2), \ldots\ ]$.

Trace Functions

A function $T : [0,1] \cap \mathbb{Q} \to (0,1) \cap \mathbb{Q}$ is a *trace function* for the α if

$$|\alpha - q| > |\alpha - T(q)| \ .$$

Contractors

A function $F : [0,1] \cap \mathbb{Q} \to (0,1) \cap \mathbb{Q}$ is a *contractor* if we have

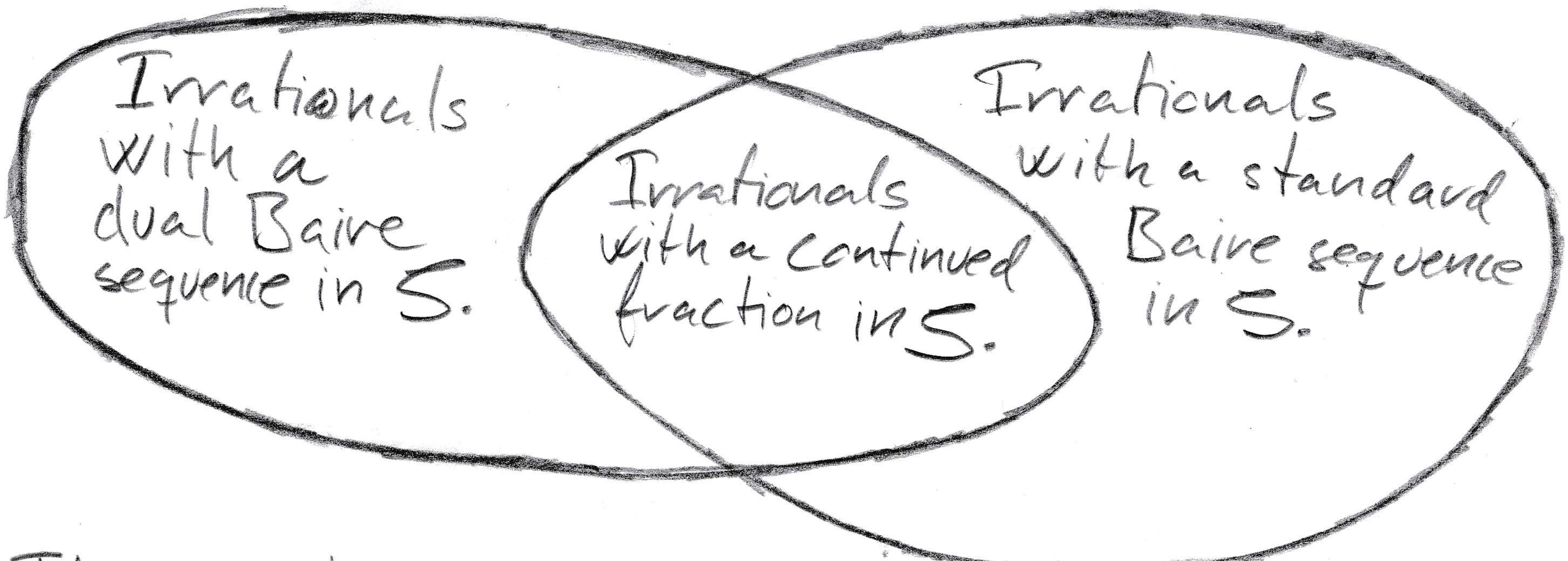
$$F(q) \neq q \quad \text{and} \quad |F(q_1) - F(q_2)| \ < \ |q_1 - q_2| \ .$$

Moreover, F is a *contractor for* α if F is a trace function for α.

Comments: It is easy to prove that any contractor is a trace function for some irrational number.

The Venn diagram at THE NEXT PAGE shows how continued fractions relate to standard and dual Baire sequences.

Let S be any subrecursive class closed under primitive recursion.



The irrationals that have a continued fraction in S are exactly those that have both a standard and a dual Baire sequence in S.

Some References

This talk is based on material published in [1, 2, 3, 4] and some unpublished material.

Subjects related to this talk been studied over the last seven decades. In a very early paper on computable analysis, Specker [17] proves that

$$\mathcal{S}_D \subset \mathcal{S}_{10E} \subset \mathcal{S}_C$$

where $\mathcal{S}$ is the class of primitive recursive functions, $\mathcal{S}_{10E}$ is the set of irrationals that have a primitive recursive decimal expansion, $\mathcal{S}_D$ is the set of irrationals that have a primitive recursive Dedekind cut and $\mathcal{S}_C$ is the set of irrationals that have a primitive recursive Cauchy sequence (Specker sequences were introduced in the same paper). In addition to Specker's paper there are works by Mostowski [15], Lehman [14], Ko [11, 12], Labhalla & Lombardi [13], Weihrauch [18], Skordev et al. [16], Georgiev [9] and quite a few more.

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