SJT- reducibility and its equivalents

André Nies
joint work with
Greenberg and Turetsky

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THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

Main Def

SJT-reducibility was introduced in Exercise of my 2009 book. It weakens $\leq_T$.

Definition (Main: SJT-reducibility)

- For an oracle B, a B–c.e. trace is a u.c.e. in B sequence $\langle T_n \rangle_{n \in \mathbb{N}}$ of finite sets.
- For a function h, such a trace is h-bounded if $|T_n| \leq h(n)$ for each n.
- A set A is jump-traceable if there is a computably bounded $\emptyset$–c.e. trace $\langle T_n \rangle_{n \in \mathbb{N}}$ such that $J^A(n)$ is in T_n if it is defined.

For sets A, B, we write $A \leq_{SJR} B$ if for each order function h, there is a B–c.e., h-bounded trace for J^A.

This is transitive by an easy argument.

Proposition

For each K-trivial set B, there is a c.e. K-trivial set A such that $A \not\leq_{SJR} B$.

A Demuth test is a seqence $\langle G_m \rangle_{m \in \mathbb{N}}$ of open subsets of $\{0,1\}^{\mathbb{N}}$ such that $\lambda G_m \leq 2^{-m}$ and there is an ω-c.a. function p such that $G_m = [W_{p(m)}]^{\prec}$.

Definition (Weak Demuth randomness)

- A nested Demuth test is a Demuth test $\langle G_m \rangle_{m \in \mathbb{N}}$ such that $G_m \supseteq G_{m+1}$ for each m.
- One says that Z is weakly Demuth random if $Z \notin \bigcap_m G_m$ for each nested Demuth test $\langle G_m \rangle_{m \in \mathbb{N}}$.

A cost function is a computable functions $c : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$. $c(x, s)$ is the cost of changing $A(x)$ at stage s. (A_s) satisfies c if the total of changes is finite.

Definition

Let c be a cost function. For sets A, B, we write $A \models_B c$ if there is a B-computable enumeration of $\langle A_s \rangle$ satisfying c.

Definition (Benign cost functions)

A cost function $\mathbf{c}$ is benign if from a rational $\epsilon > 0$, we can compute a bound on the length of any sequence $n_1 < s_1 \leq n_2 < s_2 \leq \cdots \leq n_\ell < s_\ell$ such that $\mathbf{c}(n_i, s_i) \geq \epsilon$ for all $i \leq \ell$. For example, $\mathbf{c}_\Omega$ is benign, with the bound being $1/\epsilon$.

Theorem (see Logic Blog, 2020)

The following are equivalent for K-trivial c.e. sets A, B.

(a) $A \leq_{SJR} B$

(b) $A \models_B \mathbf{c}$ *for every benign cost function* $\mathbf{c}$

(c) $A \leq_T B \oplus Y$ *for each Martin-Löf-random set* Y *that is not weakly Demuth random*

(d) $A \leq_T B \oplus Y$ *for each Martin-Löf-random set* $Y \in \mathcal{C}$, *where* $\mathcal{C}$ *is the class of the* ω*-c.a., superlow, or superhigh sets.*