Weak reducibilities on the K-trivial sets

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mostly based on joint work with
Greenberg, Miller, and Turetsky, in various combinations

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Martin-Löf randomness

A central algorithmic randomness notion for infinite bit sequences is the one of Martin-Löf. There are several equivalent ways to define it. Here is one.

$Z \in 2^{\mathbb{N}}$ is Martin-Löf random $\iff$

for every computable sequence $(\sigma_i)_{i\in\mathbb{N}}$ of binary strings with $\sum_i 2^{-|\sigma_i|} < \infty$, there are only finitely many i such that σ_i is an initial segment of Z.

ML-random sequences satisfy properties one would intuitively expect: e.g. noncomputable, law of large numbers, ...

What does a ML-random compute?

- The Kučera-Gács theorem says that each set $A \subseteq \mathbb{N}$ is Turing below some Martin-Löf random Z.
- If A is Δ^0_2, we can take Chaitin's Ω because $\Omega \equiv_T \emptyset'$

Conversely, if we are given a ML-random, which sets are Turing below it?

Theorem (Kučera 1985)

Each Δ^0_2 ML-random has a noncomputable c.e. set Turing below it.

The randomness enhancement principle (N. 2010)

The less a ML-random Z computes, the more random it gets.

Example: Z is called weakly 2-random if Z is in no null Π^0_2 class. This is stronger than ML-randomness.

Weak 2-random $\iff$ ML-random and forms a minimal pair with $\emptyset'$.

These results suggest a spectrum of randomness strength:

- from ML-random (including examples such as Ω that computes all Δ^0_2 sets)
- to weakly 2-random (computing none but the computable Δ^0_2 sets).

Enter the K-trivials

K is not the halting problem, but rather $K(x)$ denotes the descriptive complexity of a string x with respect to a universal prefix-free machine. Recall the Schnorr-Levin theorem:

- $Z \in 2^{\mathbb{N}}$ is ML-random if and only if $K(Z \upharpoonright n) \geq^{+} n$.

In the other extreme,

Definition (Chaitin, 1975)

$A \in 2^{\mathbb{N}}$ is K-trivial if $K(A \upharpoonright n) \leq^{+} K(n)$.

- computable $\Rightarrow$ K-trivial
- Chaitin: all K-trivials are Δ^0_2
- Solovay, '75: there is a noncomputable K-trivial set.

Letters A, B denote K-trivials. Letters Y, Z denote ML-randoms.

Characterisations of K-trivials

Theorem (Nies-Hirschfeldt;Nies 2003)

The following are equivalent for $A \in 2^{\mathbb{N}}$:

- A is K-trivial.
- $K^A =^+ K$ (A is low for K).
- $\mathsf{MLR}^A = \mathsf{MLR}$ (A is low for ML-randomness).

(MLR denotes the class of Martin-Löf-random infinite bit sequences.)

Theorem (Nies 2003)

- K-triviality is Turing-invariant.
- The K-trivial Turing degrees form an ideal contained in the superlow sets.
- Every K-trivial set is Trump below a c.e. K-trivial set.

Basis for randomness

Theorem (Hirschfeldt, Nies, Stephan, 2006)

$A \in 2^{\mathbb{N}}$ is K-trivial if and only if $A \leq_T Z$ for some $Z \in \mathsf{MLR}^A$.

Left to right follows from the equivalence of K-triviality with lowness for ML-randomness, and the Kučera-Gacs Theorem.

Proposition (Hirschfeldt, Nies, Stephan, 2006)

If $A \leq_T Z$ where A is c.e. and Z is ML-random with $\emptyset' \not\leq_T Z$, then $Z \in \mathsf{MLR}^A$. And hence A is K-trivial.

- In other words, if A is c.e. and NOT K-trivial, then any ML-random $Z \geq_T A$ is above $\emptyset'$.
- So there is no version of Kučera-Gacs within the Turing incomplete sets.

Characterising the c.e. K-trivials in terms of plain ML-randomness and computability notions

The converse was asked by Stephan (2006): is every (c.e.) K-trivial below an incomplete ML-random?

Theorem (Bienvenu, Greenberg, Kucera, N., Turetsky '16 & Day, Miller, '16)

The following are equivalent for a c.e. set:

- A is computable from some incomplete ML-random;

- A is K-trivial.

And in fact, there is a single incomplete Δ^0_2 ML-random above all the K-trivials!

ML-reducibility

By 2016 there were 17 or so characterisations of the class of K-trivials, but only this was known about their internal structure:

They form an ideal in the Turing degrees that is contained in superlow, generated by its c.e. members, and has no greatest degree (i.e., it is nonprincipal).

It turns out that Turing reducibility $\leq_T$ is too fine to understand the structure. A coarser "reducibility" is suggested by the results above.

Definition (main for this talk)

For sets A, B, we write $B \geq_{ML} A$ if

$$\forall Z \text{ Martin-Löf-random } \ [Z \geq_T B \Rightarrow Z \geq_T A].$$

(Any ML-random computing B also computes A.)

Recall: $B \geq_{ML} A$ if $\forall Z \in \mathsf{MLR}[Z \geq_T B \Rightarrow Z \geq_T A]$.

- A common lowness paradigm: computational lowness means to be not overly useful as an oracle.
- $\leq_{LR}$ and other weak reducibilities are based on this: quantify the usefulness of the oracle. $\leq_{SJT}$ on the last three slides also is an instance of this paradigm.
- ML-reducibility descends from an alternative lowness paradigm: computational lowness means being computed by many oracles.

By HiNiSt 06, the ML-degree of $\emptyset'$ contains all the non-K-trivial c.e. sets. So among the c.e. sets one can focus on K-trivials.

ML degrees are essentially c.e.

- Each K-trivial A is ML-equivalent to a c.e. K-trivial $D \geq_T A$. (GrMiNiTu, arXiv 1707.00258)

Structure of the K-trivials w.r.t. $\leq_{ML}$

- The least degree consists of the computable sets. This follows from the low basis theorem with upper cone avoiding.
- There is a ML-complete K-trivial c.e. set S, called a "smart" K-trivial. (BiGrKuNiTu, JEMS 2016)
- There is a dense hierarchy of principal ideals $\mathcal{B}_q$, $q \in (0,1)_{\mathbb{Q}}$. E.g., $\mathcal{B}_{0.5}$ consists of the sets that are computed by both "halves" of a ML-random Z, namely Z_{even} and Z_{odd} (GrMiNi, JML 2019)
- several other interesting subclasses of the K-trivials are downward closed under $\leq_{ML}$.
- E.g. the strongly jump traceable sets, or equivalently, the sets below all the ω-c.a. ML-randoms (by HiGrNi, Adv. Maths 2012, along with GrMiNiTu).

Degree theory for $\leq_{ML}$ on the K-trivials

Recall: $B \geq_{ML} A$ if $\forall Z \in \mathsf{MLR}[Z \geq_T B \Rightarrow Z \geq_T A]$.

Results from GrMiNiTu, arxiv 1707.00258

(a) For each noncomputable c.e. K-trivial D there are c.e. $A, B \leq_T D$ such that $A \mid_{ML} B$.

(b) There are no $\leq_{ML}$-minimal pairs among the c.e. K-trivials.

(c) For each c.e. A there is a c.e. $B >_T A$ such that $B \equiv_{ML} A$.

(a) is based on Kučera's method. (b) and (c) use cost functions.

How many random sets are needed in the definition of $\geq_{ML}$?

- Fix notation η for a computable ordinal. There's a noncomputable c.e. set D below all the η-c.a. randoms as they form a null Σ^0_3 class.
- By (a), restricting Z in the definition of $\geq_{ML}$ to the η-c.a. sets yields a weaker reducibility,

Cost functions

Definition

A cost function is a computable function $\mathbf{c}\colon \mathbb{N}^2 \to \mathbb{R}^{\geq 0}$ satisfying:

- monotonicity $\mathbf{c}(x,s) \geq \mathbf{c}(x+1,s)$ and $\mathbf{c}(x,s) \leq \mathbf{c}(x,s+1)$
- $\underline{\mathbf{c}}(x) := \lim_s \mathbf{c}(x,s) < \infty$ exists, and $\lim_x \underline{\mathbf{c}}(x) = 0$.

Definition

Let $\langle A_s \rangle$ be a computable approximation of a Δ^0_2 set A; let $\mathbf{c}$ be a cost function. The total cost $\mathbf{c}(\langle A_s \rangle)$ is

$$\sum_s \mathbf{c}(x,s)[\![x \text{ is least s.t. } A_s(x) \neq A_{s-1}(x)]\!].$$

A Δ^0_2 set A obeys a cost function $\mathbf{c}$ if there is some computable approximation $\langle A_s \rangle$ of A for which the total cost $\mathbf{c}(\langle A_s \rangle)$ is finite.

Write $A \models \mathbf{c}$ for this. FACT: There is a noncomputable c.e. $A \models \mathbf{c}$.

Cost functions characterising ML-ideals

Recall: a Δ^0_2 set obeys $\mathbf{c}$ if it can be computably approximated obeying the "speed limit" given by $\mathbf{c}$.

Let $\mathbf{c}_\Omega(x, s) = \Omega_s - \Omega_x$ (where $\langle \Omega_s \rangle$ is an increasing approximation of Ω).

Theorem (N., Calculus of cost functions, 2017)

A Δ^0_2 set is K-trivial if and only if it obeys $\mathbf{c}_\Omega$.

Let $\mathbf{c}_{\Omega,1/2}(x, s) = (\Omega_s - \Omega_x)^{1/2}$.

Theorem (GrMiNi, 2019)

The following are equivalent:

1. A is computed by both halves of a ML-random.
2. A obeys $\mathbf{c}_{\Omega,1/2}$.

Cost functions and computing from randoms

λ denotes the uniform product measure on $\{0,1\}^{\mathbb{N}}$.

Definition

Let $\mathbf{c}$ be a cost function. A $\mathbf{c}$-test is a sequence (U_n) of uniformly Σ^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ satisfying $\lambda(U_n) = O(\underline{\mathbf{c}}(n))$.

Main Fact

If $Z \in \mathsf{MLR}$ fails a $\mathbf{c}$-test, and $A \models \mathbf{c}$, then $A \leq_T Z$.

- Collect the oracles that may become invalid through A-change into a Solovay test.
- If $A_{s-1}(n) \neq A_s(n)$, then $U_{n,s}$ is listed as a component of the test. Solovay because $\langle A_s \rangle$ obeys $\mathbf{c}$.
- Z is outside almost all components, so Z computes A correctly a.e.

Definition (ML-completeness for a cost function, GrMiNiTu)

Let $\mathbf{c} \geq \mathbf{c}_\Omega$ be a cost function. We say A is ML-complete for $\mathbf{c}$ if $A \models \mathbf{c}$, and $B \leq_{ML} A$ for each $B \models \mathbf{c}$.

Theorem (GrMiNiTu, extending BiGrKuNiTu, 2016 result for $\mathbf{c}_\Omega$)

For each $\mathbf{c} \geq \mathbf{c}_\Omega$ some c.e. set A is ML-complete for $\mathbf{c}$.

Proof idea: Let Γ be the Turing functional such that $\Gamma(0^e 1 Z) = \Phi_e(Z)$. Build $A \models \mathbf{c}$ such that $A = \Gamma^Y \Rightarrow Y$ fails some $\mathbf{c}$-test. Hence $B \models \mathbf{c}$ implies $B \leq_T Y$. We may that assume $\mathbf{c}(k) \geq 2^{-k}$ for each k.

- During the construction, let $\mathcal{G}_{k,s} = \{Y : \Gamma^Y_t \restriction 2^{k+1} \prec A_t \text{ for some } k \leq t \leq s\}$.
- Error set $\mathcal{E}_s$: those Y such that Γ^Y_s is to the left of A_s.
- Ensure $\lambda \mathcal{G}_{k,s} \leq \mathbf{c}(k,s) + \lambda(\mathcal{E}_s - \mathcal{E}_k)$. If this threatens to fail, put the next $x \in [2^k, 2^{k+1})$ into A. $\langle \mathcal{G}_k \rangle$ is the required $\mathbf{c}$-test: $\lambda \mathcal{G}_k \leq \mathbf{c}(k)$.

ML-completeness for a cost function

Clearly the bigger a cost function, the harder it is to obey.

Theorem (GrMiNiTu)

For each K-trivial A there is a cost function $\mathbf{c}_{\langle A \rangle} \geq \mathbf{c}_{\Omega}$ such that A is ML-complete for $\mathbf{c}_{\langle A \rangle}$.

This shows that there are no ML-minimal pairs: suppose K-trivial sets A, B are noncomputable.

- There is a noncomputable c.e. D such that $D \models \mathbf{c}_{\langle A \rangle} + \mathbf{c}_{\langle B \rangle}$.
- Then $D \leq_{ML} A, B$.

ML completeness for cost functions, and half-bases

A is a half-base if $A \leq_T \Omega_{even}, \Omega_{odd}$.

Theorem (BiGrKuNiTu, 2016)

Not every K-trivial is a half-base.

Proof.

- Ω_{even} and Ω_{odd} are low by van Lambalgen and [HiNiSt:06];
- If $Y \in \mathsf{MLR}$ fails a $\mathbf{c}_\Omega$-test, then it is (super)high.
- So an ML-complete K-trivial has only high ML-randoms above, and hence it is not a half-base.

$\square$

A reducibility dual to $\leq_{ML}$

Definition

For $Z, Y \in \mathsf{MLR}$, let $Z \leq_{ML^*} Y$ if for every K-trivial A,

$$A \leq_T Z \quad \Rightarrow \quad A \leq_T Y.$$

- Top degree: all randoms failing a $\mathbf{c}_\Omega$-test (ie, the ML-random that are non Oberwolfach-random).
- Bottom degree: the weakly 2-randoms.

We say that $Z \in \mathsf{MLR}$ is feeble for $\mathbf{c}$ if Z fails a $\mathbf{c}$-test, and has least ML^*-degree among those.
For example: For rational $p \in (0, 1)$, any appropriate "p-part" of Ω is feeble for $\mathbf{c}_{\Omega,p}$.

Pieces of Ω w.r.t. $\leq_{ML^*}$

- For any infinite computable $R \subseteq \mathbb{N}$, let Ω_R be the bits of Ω with position in R.
- We can define a corresponding cost function $\mathbf{c}_{\Omega,R}$ similar to $\mathbf{c}_{\Omega,p}$: A obeys $\mathbf{c}_{\Omega,R} \iff A \leq_T \Omega_R$.
- Thus, Ω_R is feeble for $\mathbf{c}_{\Omega,R}$.

For each R, let B_R be a K-trivial that is ML-complete for $\mathbf{c}_{\Omega_R}$.

Theorem (GrMiNiTu, submitted)

The following are equivalent for infinite, computable $R, S \subseteq \mathbb{N}$:

1. $\Omega_S \leq_{ML^*} \Omega_R$;
2. $B_S \geq_{ML} B_R$;
3. $|S \cap n| \leq^+ |R \cap n|$.

For instance, by (3), Ω_{even} and Ω_{odd} compute the same K-trivials!

Other weak reducibilities

- Note that $A \leq_T B$ if $J^A = \Psi^B$ for some functional Ψ (where $J^X = \phi_e^X(e)$ is the jump of X).
- Suppose that B instead can only make a small number of guesses for $J^A(x)$:

Definition (N. 2009; related to Cole and Simpson 06)

We write $A \leq_{SJT} B$ if for each order function h, there is a uniform list $\langle \Psi_r \rangle$ of functionals such that $J^A(x)$, if defined, equals $\Psi_r^B(x)$ for some $r \leq h(x)$.

- This relation is weaker than Turing, and transitive.
- A is strongly jump traceable (FiNiSt 05) if $A \leq_{SJT} \emptyset$. These sets are properly contained in the K-trivials.
- There is no $\leq_{SJT}$-largest K-trivial, essentially by relativizing this.

Recall that Y is ω-c.a. if $Y \leq_{\mathrm{wtt}} \emptyset'$.

Let $\mathcal{C}$ be the class of the ω-c.a., superlow, or superhigh sets.

Theorem (with Greenberg and Turetsky)

The following are equivalent for K-trivial c.e. sets A, B.

(a) $A \leq_{SJT} B$

(b) $A \leq_T B \oplus Y$ for each $Y \in \mathcal{C} \cap \mathsf{MLR}$.

This generalises work of [GHN 2012] where $B = \emptyset$. As a corollary, all the ideals $\mathcal{B}_q$ are downward closed under $\leq_{SJT}$, because $\mathcal{B}_q$ consists of the K-trivials below an appropriate piece Ω_R of Ω, which is $\omega - c.a.$.

We have on the K-trivials that

$$\leq_T \quad \Rightarrow \quad \leq_{ML} \quad \Rightarrow \quad \leq_{\omega-\text{c.a.}-ML}$$

$$\leq_T \quad \Rightarrow \quad \leq_{SJT} \quad \Rightarrow \quad \leq_{\omega-\text{c.a.}-ML}$$

and none of $\leq_{ML}$, $\leq_{SJT}$ implies the other.

Questions

- Is being an ML-complete K-trivial an arithmetical property? Stronger: is $\leq_{\mathrm{ML}}$ an arithmetical relation?
- Are the ML-degrees of the (c.e.) K-trivials dense? (Downward density is known.)
- Can a smart K-trivial be half of a minimal pair in the c.e. Turing degrees?
- Can it obey a cost function stronger than $\mathbf{c}_\Omega$?
- Is there an incomplete ω-c.a. ML-random above all the K-trivials?

Some references

- Bienvenu, Greenberg, Kučera, Nies, Turetsky: Coherent randomness tests and computing the K-trivial sets, J. Eur. Math. Soc. 2016
- Greenberg, J. Miller, Nies: Computing from projections of random points, J. Math. Logic 2019
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