Structures Realised by Positive Equivalence Relations

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Positive Equivalence Relations

Definition. An equivalence relation is called positive iff it is an r.e. equivalence relation on $\mathbb{N}$ and the set of equivalent pairs is recursively enumerable; furthermore, it is required that there are infinitely many equivalence classes.

A subset $\mathbf{A} \subseteq \mathbb{N}$ is called $\mathbf{E}$-closed iff whenever $\mathbf{x} \in \mathbf{A}$ and $\mathbf{x}\,\mathbf{E}\,\mathbf{y}$ implies $\mathbf{y} \in \mathbf{A}$. $\mathbf{E}$-closed sets consisting of finitely many equivalence classes are called $\mathbf{E}$-finite and an r.e. $\mathbf{E}$-closed sets consisting of infinitely many equivalence classes are called $\mathbf{E}$-infinite. The $\mathbf{E}$-closure of a finite set is always $\mathbf{E}$-finite.

$\mathbf{E}$ is many-one reducible to $\mathbf{F}$ iff there is a recursive function $\mathbf{f}$ such that all $\mathbf{x}, \mathbf{y}$ satisfy $\mathbf{x}\,\mathbf{E}\,\mathbf{y} \Leftrightarrow \mathbf{f}(\mathbf{x})\,\mathbf{F}\,\mathbf{f}(\mathbf{y})$.

Examples

If $\mathbf{A}$ is a nonempty coinfinite r.e. set then the equivalence relation $\mathbf{E}$ given as $\mathbf{x\,E\,y} \Leftrightarrow \mathbf{x} = \mathbf{y} \vee (\mathbf{x} \in \mathbf{A} \wedge \mathbf{y} \in \mathbf{A})$. If $\mathbf{A}$ is simple then this has only $\mathbf{E}$-finite and $\mathbf{E}$-cofinite sets and a set is $\mathbf{E}$-cofinite iff it is a cofinite superset of $\mathbf{A}$. [Ershov]

There is a theory of complete and precomplete positive equivalence relations; complete ones allow to many-one reduce every further equivalence relation to them.

All recursive positive equivalence relations are many-one equivalent.

Recall that $\mathbf{E}$-finite and $\mathbf{E}$-infinite sets have to be both, $\mathbf{E}$-closed and recursively enumerable. Some positive equivalence relation $\mathbf{E}$ has only one $\mathbf{E}$-infinite set, namely $\mathbb{N}$. These equivalence relations are neither recursive nor complete, also their equivalence classes are never recursive.

Realising Structures

Definition. A group G is realised by an equivalence relation E iff one can identify the group members with equivalence classes in a way that there is a recursive function f which maps two representatives x, y of two group elements a, b to a representative $f(x, y)$ of $a \circ b$; similarly for other algebraic structures.

Example. Boone and Novikov (independently of each other) constructed around 1955 finitely presented groups such that their word problem was only a positive but not recursive equivalence relation (on the set of all words over fixed generators). Thus there are groups which can be realised by some positive equivalence relation E but not by a recursive one.

Relations. A structure which has a relation R requires that the relation R respected E, that is, if $x\,E\,x'$ and $y\,E\,y'$ then $x\,R\,y$ if and only if $x'\,R\,y'$. Similarly for functions.

Recursive Equivalence Relations

Definition. A (countable) algebra has domain $\mathbb{N}$ and is given by several functions from $\mathbb{N}^{\mathbf{k}} \rightarrow \mathbb{N}$.

Theorem. The following structures are only realised by recursive positive equivalence relations (if at all):
(a) All finitely generated algebras where all nontrivial quotients are finite, in particular arithmetic $(\mathbb{N}, +, \cdot)$, $(\mathbb{N}, \mathbf{Succ})$ [Malcev 1961].
(b) All countable fields and simple groups [Ershov and Goncharov 2000].
(c) All finitely presentable locally finite algebras, where finitely presentable = finitely generated with finitely many defining equations, locally finite = every two distinct $\mathbf{x}, \mathbf{y}$ are mapped to different elements by some homomorphism onto a finite structure [Malcev 1961 and MacKenzie].

Algebra-Degrees

Definition. For positive equivalence relations $\mathbf{E}$ and $\mathbf{F}$, $\mathbf{E}$ is algebra-reducible to $\mathbf{F}$ iff every algebra realised by $\mathbf{E}$ is also realised by $\mathbf{F}$.

Theorem [Gavryushkin, Khoussainov, Stephan 2016]. There are positive equivalence relations $\mathbf{E}$ which realise exactly those algebras where all defining functions are either projections or constant functions; these $\mathbf{E}$ form the least algebra-degree.

Theorem [Khoussainov 2016]. Every $\mathbf{E}$ which realises some finitely generated algebra has maximal algebra-degree.

Theorem [Khoussainov, Lempp and Slaman 2005]. There is a maximal algebra-degree which does not contain any finitely generated algebra.

Permutation Algebras

Definition. A permutation algebra consists of a bijection $\mathbf{f}$ from $\mathbb{N}$ to $\mathbb{N}$. $\mathbf{E}$ is perm-reducible to $\mathbf{F}$ iff every permutation algebra realised by $\mathbf{E}$ is also realised by $\mathbf{F}$.

Theorem [Gavryushkin, Khoussainov and Stephan 2016].
(a) The recursive positive equivalence relations form a single maximal perm-degree.
(b) If $\mathbf{E}$ is many-one equivalent to $\mathbf{F}$ via a recursive permutation then $\mathbf{E}$ is perm-equivalent to $\mathbf{F}$; however, if only $\mathbf{E} \leq_\mathbf{m} \mathbf{F}$ and $\mathbf{F} \leq_\mathbf{m} \mathbf{E}$ then it might be that $\mathbf{E}$ is not perm-equivalent to $\mathbf{F}$.
(c) There are minimal pairs of perm-degrees.
(d) There is an ascending chain of many-one degrees which is at the same time an ascending chain of perm-degrees.

Boolean Algebras

Definition [Badaev 1991]. A positive equivalence relation $\mathbf{E}$ is precomplete iff for every recursive function $\mathbf{f}$ there is an $\mathbf{x}$ with $\mathbf{x}\,\mathbf{E}\,\mathbf{f}(\mathbf{x})$.

Theorem [Bazhenov, Mustafa, Stephan and Yamaleev 2017].
(a) If $\mathbf{E}$ realises a Boolean algebra then all its equivalence classes are uniformly many-one equivalent.
(b) Weakly precomplete positive equivalence relations do not realise any Boolean algebra.
(c) The recursive positive equivalence relations are the only ones which realise all recursive Boolean algebras, thus they form a maximal element in the Boolean-algebra-derees.

Theorem [Bazhenov, Mustafa, Stephan and Yamaleev 2017]. Some positive equivalence relation $\mathbf{E}$ satisfies (a) all equivalence classes are many-one equivalent to $\mathbf{E}$, (b) $\mathbf{E}$ realises an Abelian group, (c) $\mathbf{E}$ does not realise any Boolean algebra.

Realising Graphs

Definition. A graph $(\mathbf{V}, \mathbf{F})$ is realised by positive equivalence relation $\mathbf{E}$ iff there is a bijection from $\mathbf{V}$ to the equivalence classes of $\mathbf{E}$ such that the image of $\mathbf{F}$ is recursive enumerable.

Theorem [Malcev 1961].
A positive equivalence relation can realise the directed graph $(\mathbb{N}, \mathbf{Succ})$ iff it is recursive; the same holds for $(\mathbb{Z}, \mathbf{Succ})$.

Example. Given a finitely generated group $\mathbf{G}$ with nonrecursive word problem and generators $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$ which includes the neutral element, one considers the graph with nodes $\mathbf{G} \times \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$ and the edge relation $\{((\mathbf{v}, \mathbf{a}), (\mathbf{w}, \mathbf{b})) : \mathbf{a} = \mathbf{b} \vee \mathbf{v} \circ \mathbf{a} = \mathbf{w}\}$; this directed graph with self-loops can only be realised by some nonrecursive positive equivalence relations.

Island Graphs

An island graph is an undirected graph with possible self-loops where infinitely many nodes are not contained in any edge.

An equivalence relation $\mathbf{E}$ is island-reducible to $\mathbf{F}$ iff every island graph realised by $\mathbf{E}$ is also realised by $\mathbf{F}$. The island-degrees are the degrees of positive equivalence relations with respect to island reducibility.

Examples.
The graph which connects all pair of even numbers with an edge but which does not have edges with odd numbers as endpoints is an island graph.
Finite island graphs are those graphs where only finitely many nodes are part of an edge. Such graphs still have infinitely many nodes (the full domain $\mathbb{N}$) but they do not use most of the nodes.

Findings

Theorem [Gavryushkin, Jain, Khoussainov, Stephan 2014].
(a) There is a greatest island-degree which coincides with the complete many-one degree.
(b) If $\mathbf{E}$ is many-one reducible to $\mathbf{F}$ then $\mathbf{E}$ is island-reducible to $\mathbf{F}$, but not vice versa.
(c) If a positive equivalence-relation has only one $\mathbf{E}$-infinite set, namely $\mathbb{N}$, then it realises only finite island graphs; furthermore, every positive equivalence relation realises all finite island graphs, thus the positive equivalence relations realising exactly the finite island graphs form the least island-degree.
(d) There is only one minimal island-degree above the least island-degree and one further island-degree which is minimal above the minimal degree; the latter is the join of two incomparable island-degrees.

Example: Proof for (c)

Consider any $\mathbf{E}$ and a set $\{(\mathbf{x_1}, \mathbf{y_1}), \ldots, (\mathbf{x_k}, \mathbf{y_k})\}$ of finitely many edges. Now the corresponding relation of edges in the graph is $\{(\mathbf{x}, \mathbf{y}) : \exists \mathbf{h}\, [\mathbf{x}\, \mathbf{E}\, \mathbf{x_h} \wedge \mathbf{y}\, \mathbf{E}\, \mathbf{y_h}]\}$ which respects $\mathbf{E}$ and is generated from the edges. Thus every positive equivalence relation $\mathbf{E}$ realises all finite island graphs.

If $\mathbf{E}$ has only one $\mathbf{E}$-infinite set (namely $\mathbb{N}$) then the union of all $\mathbf{E}$-equivalence classes being an endpoint of some edge of an island graph cannot contain all equivalence classes and is $\mathbf{E}$-finite, hence there are only finitely many edges.

An $\mathbf{E}$ which has only one $\mathbf{E}$-infinite set can be constructed as follows: At each stage one has finite sets $[\mathbf{a_{0,t}}], [\mathbf{a_{1,t}}], \ldots$ which partition $\mathbb{N}$. In stage $\mathbf{t}$, find least triple $(\mathbf{m}, \mathbf{e}, \mathbf{n})$ with $\mathbf{m} + \mathbf{e} < \mathbf{n}$, $\mathbf{W_{e,t}} \cap [\mathbf{a_{m,t}}] = \emptyset$, $\mathbf{W_{e,t}} \cap [\mathbf{a_{n,t}}] \neq \emptyset$ and update $\mathbf{a_{m,t+1}} = \mathbf{a_{m,t}} \cup \mathbf{a_{n,t}}$ and $\mathbf{a_{k,t+1}} = \mathbf{a_{k+1,t}}$ for $\mathbf{k} \geq \mathbf{n}$.

Results for General Graphs

Theorem [Gavryushkin, Jain, Khoussainov, Stephan 2014].
(a) There is a least graph-degree in which every graph realised contains only finitely many edges.
(b) There is an ascending chain $\mathbf{E_0, E_1, \ldots}$ of graph-degrees such that between $\mathbf{E_k}$ and $\mathbf{E_{k+1}}$ there is no further graph-degree.
(c) There are exactly two minimal graph-degrees which are given by locally finite graphs (every node has only finitely many neighbours).
(d) There is no greatest graph-degree and every graph-degree which realises a recursively categorical graph is maximal.

Open Question. Are there infinitely many minimal graph-degrees?

Linear Orders

Definition A positive equivalence relation $\mathbf{E}$ realises a linear order $<$ iff one can identify the equivalence classes with the elements of the partially ordered set $(\mathbf{A}, <)$ such that the corresponding preorder $\leq$ is recursively enumerable and respects $\mathbf{E}$.

Remark. If $\mathbf{E}$ realises $<$ itself (instead of $\leq$) then $\mathbf{E}$ is a recursive positive equivalence relation.

Theorem [Fokina, Khoussainov, Semukhin, Turetsky 2016].
(a) There is an infinite antichain of lo-degrees.
(b) There is an infinite chain of lo-degrees.
(c) There is a maximal lo-degree which is given by the lo-degree of all recursive positive equivalence relations.
(d) The structure of lo-degrees is not an upper semilattice.

Learning Theory

Idea: Learner receives an infinite sequence of data containing all members of a recursively enumerable set (plus perhaps pause symbols) and outputs in response an infinite set of conjectures. Such sequences are called text. [Gold 1967; early follow-up work in East-Germany, Latvia, USA, Japan].

Learner $\mathbf{M}$ learns class $\mathbf{L_0}, \mathbf{L_1}, \ldots$ of r.e. languages iff for every $\mathbf{L_e}$ and every text $\mathbf{T}$ of $\mathbf{L_e}$ the hypotheses of $\mathbf{M}$ converge to a $\mathbf{L_e}$ in one of the below sense. There are various notions of learning, the basic one is explanatory by Gold (1967), others followed later.

Example. Learner sees $\mathbf{2\,2\,5\,2\,2\,5\,4\,8\,9}\ \ldots$ and conjectures $\{\mathbf{2}\}$ (twice), $\{\mathbf{2}, \mathbf{5}\}$ (four times), $\{\mathbf{2}, \mathbf{4}, \mathbf{5}\}$ (twice), $\{\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{8}\}$, $\{\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{8}, \mathbf{9}\}$, $\ldots$ and this learner converges on all finite sets to a listing of the set, but fails to learn any infinite set (like the set of all prime powers).

Learning Criteria

Formally learner does not conjecture lists of elements of sets, but indices of a family of uniformly r.e. sets containing all members of the family $\{L_0, L_1, \ldots\}$ to be learnt; let L_e be the current learning task.

(a) Behaviourally correct: Almost all indices are for L_e with respect to some underlying indexing of the r.e. languages;
(b) Vacillatory: As (a), but from some time onwards, the learner vacillates only between finitely many correct hypotheses;
(c) Explanatory: The learner outputs a sequence of hypotheses converging to one correct hypothesis;
(d) Confident: As (c), but learner converges on every text (also on those not for any L_e) to some index;
(e) Finite: The learner outputs after some special symbols $?$ one correct hypothesis which is never revised.

Examples

The class of all five-element sets can be learnt finitely.
The class of all four- and all five-element sets can be learnt confidently but not finitely.
The class of all finite sets can be learnt explanatorily but not confidently, as the learner will on some texts for infinite languages make infinitely many different hypotheses.
The class of all sets $\{e, e+1, e+2, \ldots\}$ plus all sets $\{e+d : d \leq |W_e|\}$ can be learnt vacillatorily but not explanatorily.
The class of all sets $K \cup F$ where F is finite and K is the halting problem can be learnt behaviourally correctly but not vacillatorily, the learner just conjectures the union of K and the set of all elements observed so far.

Learning and Positive Equivalence

Given positive equivalence relation $\mathbb{E}$, consider a family of $\mathbb{E}$-closed r.e. sets which has a one-one numbering. Previous slide showed that for the learning criteria considered, there is a proper hierarchy and each level contains examples. Now the question is how does this hierarchy behave with respect to the setting of $\mathbb{E}$-closed families with one-one numberings, from now on called $\mathbb{E}$-families. Results may depend on choice of $\mathbb{E}$. Recursive equivalence relations replicate those classic results which use one-one families.

Angluin initiated the study of learning from indexed families where every set in the family is recursive, this had comprehensive follow-up work. Infinite indexed families can be made one-one. However one cannot keep in the setting of $\mathbb{E}$-families that the $\mathbb{E}$-family is uniformly recursive, as then some $\mathbb{E}$ would not realise any $\mathbb{E}$-families at all.

Examples of E-Families

Order the $\mathbf{E}$-equivalence classes by their least elements $\mathbf{a_0, a_1, \ldots}$ so that $[\mathbf{a_n}]$ is the equivalence class generated by $\mathbf{a_n}$. Now the family of all $\mathbf{A_n = [\{a_0, a_1, \ldots, a_{n-1}\}]}$ is an $\mathbf{E}$-family.

If $\mathbf{f}$ is recursive and strictly increasing then every r.e. superfamily of $\mathbf{A_{f(0)}, A_{f(1)}, A_{f(2)}, \ldots}$ has a one-one numbering and is therefore an $\mathbf{E}$-family. In particular the family of all $\mathbf{E}$-finite sets is an $\mathbf{E}$-family.

There are positive equivalence relations $\mathbf{E}$ such that
(a) $\mathbf{\{[a_n] : n = 0, 1, \ldots\}}$ is not an $\mathbf{E}$-family and
(b) every $\mathbf{E}$-family contains comparable sets (one set is a proper subset of the other one).

There is an $\mathbf{E}$-family and an $\mathbf{a_n}$ not contained in any member as an element iff there are at least two $\mathbf{E}$-infinite sets.

Behaviourally Correct Learning

Theorem [Belanger, Gao, Jain, Li and Stephan 2021].
(a) All ehaviourally correctly learnable families consist only of $\mathbf{E}$-finite sets iff $\mathbb{N}$ is the only $\mathbf{E}$-infinite set.
(b) Every positive equivalence relation $\mathbf{E}$ has an $\mathbf{E}$-family which is behaviourally correctly learnable but not vacillatorily learnable.
(c) It depends on the $\mathbf{E}$-family whether vacillatory and explanatory learning coincide or not; for some explicit constructed $\mathbf{E}$ they coincide; for recursive equivalence relations, they do not coincide.

Explanatory Learning

Theorem [Belanger, Gao, Jain, Li and Stephan 2021].
(a) The $\mathbf{E}$-family $\{\mathbf{A_n} : \mathbf{n} = \mathbf{0}, \mathbf{1}, \ldots\}$ is for every positive equivalence relation $\mathbf{E}$ explanatorily learnable but not confidently learnable.
(b) There is a positive equivalence relation without $\mathbf{E}$-families which are confidently learnable; there are then also no finitely learnable $\mathbf{E}$-families.
(c) If there is an $\mathbf{E}$-family which is finitely learnable then the hierarchy Finite $\Rightarrow$ Confidently $\Rightarrow$ Explanatorily learnable is strict.
(d) There is some $\mathbf{E}$ for which there are confidently but not finitely learnable $\mathbf{E}$-families.

Example: Learner for (a)

Let $\mathbf{E_t}$ be approximation to $\mathbf{E}$ for $\mathbf{t}$ steps and let $\mathbf{a_{n,t}}$ be the $\mathbf{n}$-th number $\mathbf{k}$ (starting with zeroth number $\mathbf{0}$) such that $\neg \mathbf{h}\, \mathbf{E_t}\, \mathbf{k}$ for all $\mathbf{h} < \mathbf{k}$, $\mathbf{a_{n,t}}$ converges to $\mathbf{a_n}$.

The $\mathbf{t}$-th guess of learner is that index $\mathbf{n}$ (standing for set $\mathbf{A_n} = [\{\mathbf{a_0}, \mathbf{a_1}, \ldots, \mathbf{a_{n-1}}\}]$) such that $\mathbf{n}$ is the first number with $\mathbf{a_{n,t}}$ not appearing among the first $\mathbf{t}$ data-items observed so far.

When learning $\mathbf{A_n}$ and $\mathbf{t}$ is large enough, then $\mathbf{a_{m,t}}$ has converged to $\mathbf{a_m}$ for all $\mathbf{m} \leq \mathbf{n}$ and $\mathbf{a_0}, \mathbf{a_1}, \ldots, \mathbf{a_{n-1}}$ have shown up in the text but $\mathbf{a_n}$ not (as $\mathbf{a_n} \notin \mathbf{A_n}$), thus the learner will for all sufficiently large $\mathbf{t}$ make $\mathbf{n}$ as the guess number $\mathbf{t}$.

As $\mathbb{N}$ is the union of all $\mathbf{A_n}$, every learner of $\{\mathbf{A_0}, \mathbf{A_1}, \ldots\}$ does not converge on some text for $\mathbb{N}$ to a single index and is thus not confident.

Non-Union Theorem

In standard inductive inference there are explanatorily learnable classes whose union is not behaviourally correctly learnable. This is called the Blum and Blum non-Union Theorem.

Theorem [Belanger, Gao, Jain, Li and Stephan 2021].
(a) The non-Union Theorem holds for behaviourally correct learning iff there are at least two $\mathbf{E}$-infinite sets.
(b) There are $\mathbf{E}$-families which are explanatorily learnable such that their union is an $\mathbf{E}$-family but not explanatorily learnable.
(c) Adding a single set to a confidently learnable $\mathbf{E}$-family does not destroy confident learnability. However, if one adds $\mathbb{N}$ to the $\mathbf{E}$-family $\{\mathbf{A_n} : \mathbf{n} = \mathbf{0}, \mathbf{1}, \ldots\}$ then this $\mathbf{E}$-family is not behaviourally correctly learnable.
(d) If there is a finitely learnable $\mathbf{E}$-family then the finitely learnable $\mathbf{E}$-families are not closed under union.

Hypothesis Spaces

The hypothesis space is a numbering which is not necessarily one-one such that the hypotheses of the learners are indices in this numbering.

Lange and Zeugmann considered in a series of papers exact hypothesis spaces (which are the given $\mathbf{E}$-families and which are one-one), class-preserving hypothesis spaces (where all indices are for members of the $\mathbf{E}$-family) and class comprising hypothesis spaces.

If a behaviourally correct family is exactly behaviourally correctly learnable then it is explanatorily learnable.

There are some $\mathbf{E}$ and some $\mathbf{E}$-families which are class-preservingly explanatorily learnable but not exactly explanatorily learnable.

The $\mathbf{E}$-family $\{\mathbf{A_n} : \mathbf{n} = \mathbf{0}, \mathbf{1}, \ldots\}$ is for all $\mathbf{E}$ exactly explanatorily learnable.

Summary

Realisation-degrees compare positive equivalence relations with respect to the question which structures of a given type they can realise. The most prominent results for these topics were listed in this survey talk.

Investigations were carried out for general types of structures (all algebras) or for easy structures (island graphs, permutation algebras, ...) but detailed studies for advanced algebraic structures like groups are missed out, as they require excellent knowledge of group theory and the prior literature.

Investigations in learning theory aim on the question how the learning hierarchies and criteria interact when learning $\mathbf{E}$-families for certain positive equivalence relations $\mathbf{E}$. Which separations from learning theory stand for all $\mathbf{E}$ and which criteria do either not have a learnable $\mathbf{E}$-family at all or collapse to another criterion for certain $\mathbf{E}$.