Generalizing Besicovitch-Davis Theorem

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Pointed set

Definition

A pointed set P is a perfect set so that for every member $x \in P$, $P \leq_T x$.

TD and *sTD*

Definition

1. Turing determinacy (TD) says that for every set A of *Turing degrees*, either A or the complement of A contains an upper cone.
2. Strong Turing determinacy (sTD) says that for every set A of *reals* ranging Turing degrees cofinally, A has a pointed subset.

AD

Theorem (Martin)

Over ZF, $AD \to sTD \to TD$.

Axiom of Choice

Definition

Given a nonempty set A,

1. CC_A, the countable choice for subsets of A, says that for any countable sequence $\{A_n\}_{n\in\omega}$ of nonempty subsets of A, there is a function $f:\omega \to A$ so that $\forall n(f(n) \in A_n)$.
2. DC_A, the dependent choice for subsets of A, says that for any binary relation $R \subseteq A \times A$, if $\forall x \in A \exists y \in A R(x,y)$, there is a countable sequence elements $\{x_n\}_{n\in\omega}$ so that $\forall n R(x_n, x_{n+1})$.

Determinacy v.s. Choice

Clearly AD implies $\neg AC$.

Theorem (Mycielski)

$ZF + AD$ implies $CC_{\mathbb{R}}$.

Theorem (Kechris)

$ZF + V = L(\mathbb{R}) + AD$ implies DC.

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$ZF + V = L(\mathbb{R}) + AD$ implies DC.

Question

Does $ZF + AD$ imply $DC_{\mathbb{R}}$?

TD v.s. Choice

Theorem (Peng and Y.)

$ZF + TD$ implies $CC_{\mathbb{R}}$.

An introduction to Fractal geometry (1)

Given a non-empty $U \subseteq \mathbb{R}$, the *diameter* of U is

$$diam(U) = |U| = \sup\{|x - y| : x, y \in U\}.$$

Given any set $E \subseteq \mathbb{R}$ and $d \geq 0$, let

$$\mathcal{H}^d(E) = \lim_{\delta \to 0} \inf\{\sum_{i<\omega} |U_i|^d : \{U_i\} \text{ is an open cover of } E \wedge \forall i\, |U_i| < \delta\},$$

$$\mathcal{P}_0^d(E) = \lim_{\delta \to 0} \sup\{\sum_{i<\omega} |B_i|^d : \{B_i\}$$

is a collection of disjoint balls of radii at most δ with centres in $E\}$.

and

$$\mathcal{P}^d(E) = \inf\{\sum_{i<\omega} \mathcal{P}_0^d(E_i) \mid E \subseteq \bigcup_{i<\omega} E_i\}.$$

An introduction to Fractal geometry (2)

Definition

Given any set E,

1. the *Hausdorff dimension* of E,

$$\mathrm{Dim}_{\mathrm{H}}(E) = \inf\{d \mid \mathcal{H}^d(E) = 0\};$$

2. the *Packing dimension* of E,

$$\mathrm{Dim}_{\mathrm{P}}(E) = \inf\{d \mid \mathcal{P}^d(E) = 0\}.$$

Besicovitch-Davis and Joyce-Preiss theorem

Theorem (Besicovitch and Davis)

For any analytic set A, $\mathrm{Dim}_{\mathrm{H}}(A) = \sup_{F \subseteq A \wedge F \text{ is closed}} \mathrm{Dim}_{\mathrm{H}}(F)$.

Theorem (Joyce and Preiss)

For any analytic set A, $\mathrm{Dim}_{\mathrm{P}}(A) = \sup_{F \subseteq A \wedge F \text{ is closed}} \mathrm{Dim}_{\mathrm{H}}(F)$.

Lutz-Lutz theorem

Theorem (Lutz and Lutz)

For any set A of reals,

$$\mathrm{Dim}_{\mathrm{H}}(A) = \inf_{x\in\mathbb{R}} \sup_{y\in A} \underline{\lim}_{n\to\infty} \frac{K^x(y \upharpoonright n)}{n}$$

and

$$\mathrm{Dim}_{\mathrm{P}}(A) = \inf_{x\in\mathbb{R}} \sup_{y\in A} \overline{\lim}_{n\to\infty} \frac{K^x(y \upharpoonright n)}{n}.$$

$\underline{\lim}_{n\to\infty} \frac{K^x(y\upharpoonright n)}{n}$ is called effective Hausdorff dimension of y relative to x; and $\overline{\lim}_{n\to\infty} \frac{K^x(y\upharpoonright n)}{n}$ effective packing dimension of y relative to x.

Slaman's theorem

Theorem (Slaman)

Assume that $V = L$, then both BM- and JP-theorems fail for a Π^1_1-set.

Proof.

Suppose that $(\mathbb{R})^L$ is not null, then by Lutz-Lutz theorem, the Π^1_1 set $\mathcal{C} = \{x \mid x \in L_{\omega_1^x}\}$ has Hausdorff dimension 1. Note that every Borel subset of $\mathcal{C}$ is countable and so has packing dimension 0. □

Low for dimension

A real x is called low for (Hausdorff-, packing-)dimension if for any s and any real y with effective dimension s, then y has effective (Hausdorff-, packing-) dimension s relative to x.

Theorem (Herbert; Lempp, Miller, Ng, Turetsky and Weber)

For any real x, there is a real y low for Hausdorff dimension but $y' \geq_T x$. *The same is true for packing dimension.*

Generalizing BD- and JP- theorems

Theorem (Peng, Wu and Y; Crone, Fishman and Jackson proves the consequence under $ZF + DC + AD$.)

Assume that $ZF + DC + sTD$, BD- and JP- theorems hold for every set of reals.

Proof.

Fix any nonempty set A. For the simplicity, we may assume that $Dim_H(A) = 1$.
By the results above, there is some e so that
$B = \{y \mid \Phi_e^{y'} \in A$ has effective Hausdorff dimension 1 relative to $y\}$ ranges Turing degrees cofinally. By sTD, B has a pointed subset P. Then $C = \{r \mid \exists y \in P\Phi_e^{y'} = r\}$ is an analytic subset of A with Hausdorff dimension 1. □

LK-reduction

Definition

$x \leq_{LK} y$ if $\overline{\lim}_{n\to+\infty} K^x(n) - K^y(n) < +\infty$.

Building a LK-powerful real (1)

Let $(P, \leq)$ be a partial order so that

- $P = \{(\sigma, F) \mid \sigma \in \omega^{<\omega} \wedge F$ is a finte subset of reals $\wedge$ $\sum_{x \in F} \sum_{n \geq |\sigma|} 2^{-K^x(n)} + \sum_{n < |\sigma|} 2^{-\sigma(n)} < 1\}$;
- $(\tau, F_0) \leq (\sigma, F_1)$ if $\tau \succeq \sigma$, and $F_1 \subseteq F_0$ and $\forall n \in |\tau| - |\sigma| \forall x \in F_1 (\tau(n) \leq K^x(n))$.

By Lutz-Lutz theorem and the forcing above, for any set A, if g is sufficiently generic, then

$$\mathrm{Dim}_{\mathrm{H}}(A) = \sup_{y \in A} \underline{\lim}_{n \to \infty} \frac{K^g(y \upharpoonright n)}{n}$$

and

$$\mathrm{Dim}_{\mathrm{P}}(A) = \sup_{y \in A} \overline{\lim}_{n \to \infty} \frac{K^g(y \upharpoonright n)}{n}.$$

Building a LK-powerful real (2)

$(P, \leq)$ is a c.c.c. forcing. To see this, it suffices to show that $(\sigma, F_0) \in P$ and $(\sigma, F_1) \in P$ are compatible. There must be some very large $n > |\sigma|$ so that $\sum_{i<|\sigma|} 2^{-\sigma(i)} + \sum_{|\sigma|\leq i<n} 2^{-\min\{K^x(i) \mid x\in F_0\cup F_1\}} + \sum_{x\in F_0\cup F_1} \sum_{m\geq n} 2^{-K^x(m)} < 1$. Let τ extending σ of length n so that for any $i \in [|\sigma|, n)$, $\tau(i) = \min\{K^x(i) \mid x \in F_0 \cup F_1\}$. Then $(\tau, F_0 \cup F_1)$ is a stronger extension.

Martin's axiom

Marin's axiom says that for any c.c.c. forcing $(\mathbf{P}, \leq)$, any $\kappa < 2^{\aleph_0}$ and any collection $\{D_\alpha\}_{\alpha<\kappa}$ of dense subsets of $\mathbf{P}$, there is a filter G meeting D_α for every $\alpha < \kappa$.

Lemma

Assume $ZFC + MA + \neg CH$, for any sequence reals $\{x_\alpha\}_{\alpha<\kappa}$ with $\kappa < 2^{\aleph_0}$, there is a real g so that $\forall \alpha < \kappa (x_\alpha <_{LK} g)$.

A powerful real

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The key lemma (1)

Lemma

Assume ZFC + MA + ¬CH, for any sequence sets of reals $\{A_\alpha\}_{\alpha<\kappa}$ *with* $\kappa < 2^{\aleph_0}$, $\mathrm{Dim_H}(\bigcup_{\alpha<\kappa} A_\alpha) = \sup_{\alpha<\kappa} \mathrm{Dim_H}(A_\alpha)$ *and* $\mathrm{Dim_P}(\bigcup_{\alpha<\kappa} A_\alpha) = \sup_{\alpha<\kappa} \mathrm{Dim_P}(A_\alpha)$.

Proof.

For any $\alpha < \kappa$, there is a real g_α so that

$$\mathrm{Dim_H}(A_\alpha) = \sup_{y \in A_\alpha} \underline{\lim}_{n\to\infty} \frac{K^{g_\alpha}(y \upharpoonright n)}{n}$$

and

$$\mathrm{Dim_P}(A_\alpha) = \sup_{y \in A_\alpha} \overline{\lim}_{n\to\infty} \frac{K^{g_\alpha}(y \upharpoonright n)}{n}.$$

The key lemma (2)

Proof.

By MA, there is a real g so that $g_\alpha <_{LK} g$ for any $\alpha < \kappa$. Then

$$\mathrm{Dim}_{\mathrm{H}}(\bigcup_{\alpha<\kappa} A_\alpha) = \sup_{y\in\bigcup_{\alpha<\kappa} A_\alpha} \underline{\lim}_{n\to\infty} \frac{K^g(y \upharpoonright n)}{n}$$

and

$$\mathrm{Dim}_{\mathrm{P}}(\bigcup_{\alpha<\kappa} A_\alpha) = \sup_{y\in\bigcup_{\alpha<\kappa} A_\alpha} \overline{\lim}_{n\to\infty} \frac{K^g(y \upharpoonright n)}{n}.$$

So the lemma follows. □

Sierpinski's theorem

Theorem (Sierpinski)

Every Σ^1_2-set is a union of $\aleph_1$-many Borel sets.

The proof

Proof.

By putting previous results together, for any Σ^1_2-set A, there is a countable sequence Borel subsets $\{A_n\}_{n\in\omega}$ of A so that

$$\mathrm{Dim}_{\mathrm{H}}(A) = \mathrm{Dim}_{\mathrm{H}}(\bigcup_n A_n)$$

and

$$\mathrm{Dim}_{\mathrm{P}}(A) = \mathrm{Dim}_{\mathrm{P}}(\bigcup_n A_n).$$

Then by BD- and JP- theorem for Borel sets. □

Some questions

Question

1. *What is the exact consistency strength of BD- and JP- theorems?*
2. *What is the consistence strength of the statement "$ZF + DC +$ every non-null set has a perfect subset"?*

谢谢