Discovering structure within the class of K-trivial sets

André Nies
mostly based on joint work with
Greenberg, Miller, Turetsky, in various combinations

Feb 3, 2022

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

Martin-Löf randomness

A central algorithmic randomness notion for infinite bit sequences is the one of Martin-Löf.
There are several equivalent ways to define it. Here is one.

$Z \in 2^{\mathbb{N}}$ is Martin-Löf random $\iff$

for every computable sequence $(\sigma_i)_{i\in\mathbb{N}}$ of binary strings with $\sum_i 2^{-|\sigma_i|} < \infty$, there are only finitely many i such that σ_i is an initial segment of Z.

ML-random sequences satisfy properties one would intuitively expect fro randomness: noncomputable, law of large numbers, ...

What can ML-random oracles compute?

- The Kučera-Gacs theorem says that each set $A \subseteq \mathbb{N}$ is Turing below some ML-random Z.
- If A is Δ^0_2, we can let $Z =$ Chaitin's Ω because $\Omega \equiv_T \emptyset'$

Conversely, if we are given a ML-random, which sets are Turing below it?

Theorem (Kučera 1985)

Each Δ^0_2 ML-random has a noncomputable c.e. set Turing below it.

Notation: ML stands for Martin-Löf.

MLR is the class of ML-random infinite bit sequences.

The randomness enhancement principle (N. 2010)

The less a ML-random Z computes, the more random it gets.

Example: Z is called weakly 2-random if Z is in no null Π^0_2 class. This is stronger than ML-randomness.

Weak 2-random $\iff$ ML-random and forms a minimal pair with $\emptyset'$.

These results suggest a spectrum of randomness strength:

- from ML-random (including examples such as Ω that computes all Δ^0_2 sets)
- to weakly 2-random (computing none but the computable sets).

Enter the K-trivials

Recall the Schnorr-Levin theorem:

- $Z \in 2^{\mathbb{N}}$ is ML-random if and only if $K(Z \upharpoonright n) \geq^{+} n$.

In the other extreme,

Definition (Chaitin, 1975)

$A \in 2^{\mathbb{N}}$ is K-trivial if $K(A \upharpoonright n) \leq^{+} K(n)$.

- computable $\Rightarrow$ K-trivial
- Chaitin: all K-trivials are Δ^0_2
- Solovay, '75: there is a noncomputable K-trivial set.

Letters A, B denote K-trivials. Letters Y, Z denote ML-randoms.

Characterisations of K-trivials

Theorem (Nies-Hirschfeldt;Nies 2003)

The following are equivalent for $A \in 2^{\mathbb{N}}$:

1. A is K-trivial.
2. $K^A =^+ K$ (A is low for K).
3. $\mathsf{MLR}^A = \mathsf{MLR}$ (A is low for ML-randomness).

Theorem (Nies 2003)

1. K-triviality is Turing-invariant.
2. The K-trivial Turing degrees form an ideal contained in the superlow sets.
3. Every K-trivial set is Turing below a c.e. K-trivial set.

Basis for randomness

Theorem (Hirschfeldt, Nies, Stephan, 2006)

$A \in 2^{\mathbb{N}}$ is K-trivial if and only if $A \leq_T Z$ for some $Z \in \mathsf{MLR}^A$.

Left to right follows from the equivalence of K-triviality with lowness for ML-randomness, and the Kučera-Gacs Theorem.

Proposition (Hirschfeldt, Nies, Stephan, 2006)

If $A \leq_T Z$ where A is c.e. and Z is ML-random with $\emptyset' \not\leq_T Z$, then $Z \in \mathsf{MLR}^A$. And hence A is K-trivial.

- In other words, if A is c.e. and NOT K-trivial, then any ML-random $Z \geq_T A$ is above $\emptyset'$.
- So there is no version of Kučera-Gacs within the Turing incomplete sets.

Characterising the c.e. K-trivials in terms of plain ML-randomness and computability notions

We've seen that every c.e. set below a Turing incomplete ML-random is K-trivial.

Stephan (2006) asked whether the converse holds as well.

Theorem (Bienvenu, Greenberg, Kucera, N., Turetsky '16 & Day, Miller, '16)

The following are equivalent for a c.e. set:

- A is computable from some incomplete ML-random;
- A is K-trivial.

And in fact, there is a single incomplete Δ^0_2 ML-random above all the K-trivials!

Farily recent characterizations of the K-trivials

- A is K-trivial iff for each ML-random Y, the symmetric difference $Y \triangle A$ is ML-random (Kuyper and Miller, 2017)
- A is K-trivial iff for all Y such that Ω is Y-random, Ω is $Y \oplus A$-random (Greenberg, Miller, Monin, and Turetsky, 2018)

ML-reducibility

Initials of authors related to this study

- **B** Laurent Bienvenu
- **G** Noam Greenberg
- **H** Denis Hirschfeldt
- **K** Antonin Kučera
- **M** Joseph Miller
- **N** André Nies
- **T** Dan Turetsky

By 2018 there were 18 or so characterisations of the class, but only this was known about the structure of the K-trivials:

They form an ideal in the Turing degrees that is contained in the superlow degrees, generated by its c.e. members, has no greatest member, and contains SJT as a proper subidead.

It appears that Turing reducibility is too fine to understand the structure. We are looking from too close to see the structure.
A coarser "reducibility" is suggested by the results above.

Definition (main for this talk)

For sets A, B, we write $B \geq_{ML} A$ if

$$\forall Z \in \mathsf{MLR}\ [Z \geq_T B \Rightarrow Z \geq_T A].$$

(Any ML-random computing B also computes A.)

Recall: $B \geq_{ML} A$ denotes that $\forall Z \in \mathsf{MLR}[Z \geq_T B \Rightarrow Z \geq_T A]$.

- A common paradigm: computational lowness means to be not overly useful as an oracle. $\leq_{LR}$ and other weak reducibilities are based on this. Later on we will introduce $\leq_{SJT}$, a weakening of $\leq_T$, also based on this paradigm.

- ML-reducibility seeks to understand relative complexity of sets via an alternative lowness paradigm: computational lowness means being computed by many oracles.

Some results related to ML degrees

- By HNS 06, the ML-degree of $\emptyset'$ contains all the non-K-trivial c.e. sets. So among the c.e. sets one can focus on K-trivials.
- Each K-trivial A is ML-equivalent to a c.e. K-trivial $D \geq_T A$. (GMNT 22, arXiv 1707.00258)

Structure of the K-trivials w.r.t. $\leq_{ML}$

- The least degree consists of the computable sets . This follows from the low basis theorem with upper cone avoiding.
- There is a ML-complete K-trivial set, called a "smart" K-trivial. (BGKNT, JEMS 2016)
- There is a dense hierarchy of principal ideals $\mathcal{B}_q$, $q \in (0,1)_{\mathbb{Q}}$. E.g., $\mathcal{B}_{0.5}$ consists of the sets that are computed by both "halves" of a ML-random Z, namely Z_{even} and Z_{odd} (GMN, JML 2019)
- there are further interesting subclasses of the K-trivials that are downward closed under $\leq_{ML}$.
- E.g. the strongly jump traceables, or equivalently the sets below all the ω-c.a. ML-randoms (by HGN, Adv. Maths 2012, along with GMNT 22).

A bit of degree theory for $\leq_{ML}$ on the K-trivials

Recall: $B \geq_{ML} A$ if $\forall Z \in \mathsf{MLR}[Z \geq_T B \Rightarrow Z \geq_T A]$.

Results from GMNT 22, arxiv 1707.00258

(a) For each noncomputable c.e. K-trivial D there are c.e. $A, B \leq_T D$ such that $A \mid_{ML} B$.

(b) There are no minimal pairs.

(c) For each c.e. A there is c.e. $B >_T A$ such that $B \equiv_{ML} A$.

(a) is based on a method of Kučera. (b) and (c) use cost functions.

We don't know whether the ML-degrees of the K-trivials are dense. The obstacle is that we don't even know whether $\leq_{ML}$ is arithmetical. So it's hard to envisage a construction showing density.

The results also hold for the coarser, arithmetical reducibility where the ML-randoms are restricted to the Δ^0_2 sets. Density may be easier to show in this case.

Cost functions

Definition

A cost function is a computable function $\mathbf{c}\colon \mathbb{N}^2 \to \mathbb{R}^{\geq 0}$ satisfying:
$\mathbf{c}(x, s) \geq \mathbf{c}(x+1, s)$ and $\mathbf{c}(x, s) \leq \mathbf{c}(x, s+1)$;
$\underline{\mathbf{c}}(x) = \lim_s \mathbf{c}(x, s) < \infty$;
$\lim_x \underline{\mathbf{c}}(x) = 0$ (the limit condition).

Definition

Let $\langle A_s \rangle$ be a computable approximation of a Δ^0_2 set A.
Let $\mathbf{c}$ be a cost function. The total cost $\mathbf{c}(\langle A_s \rangle)$ is

$$\sum_s \mathbf{c}(x, s)[\![x \text{ is least s.t. } A_s(x) \neq A_{s-1}(x)]\!].$$

A Δ^0_2 set A obeys a cost function $\mathbf{c}$ if there is some computable approximation $\langle A_s \rangle$ of A for which the total cost $\mathbf{c}(\langle A_s \rangle)$ is finite.

Write $A \models \mathbf{c}$ for this. FACT: There is a c.e., noncomputable $A \models \mathbf{c}$.

Cost functions characterising ML-ideals

In summary, a Δ^0_2 set obeys $\mathbf{c}$ if it can be computably approximated obeying the "speed limit" given by $\mathbf{c}$.

Let $\mathbf{c}_\Omega(x,s) = \Omega_s - \Omega_x$ (where $\langle \Omega_s \rangle_{s\in\mathbb{N}}$ is an increasing approximation of Ω).

Theorem (N., Calculus of cost functions, 2017)

A Δ^0_2 set is K-trivial if and only if it obeys $\mathbf{c}_\Omega$.

Let $\mathbf{c}_{\Omega,1/2}(x,s) = (\Omega_s - \Omega_x)^{1/2}$.

Theorem (GMN, 2019)

The following are equivalent:

1. A is computed by both halves of a ML-random.
2. A obeys $\mathbf{c}_{\Omega,1/2}$.

Cost functions and computing from randoms

Definition

Let $\mathbf{c}$ be a cost function. A $\mathbf{c}$-test is a sequence (U_n) of uniformly Σ^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ satisfying $\lambda(U_n) = O(\underline{\mathbf{c}}(n))$.

Important Easy Fact

Suppose that Z is ML-random but is captured by a $\mathbf{c}$-test. Suppose that A obeys $\mathbf{c}$. Then $A \leq_T Z$.

Proof idea: Collect the oracles that may become invalid through A-change into a Solovay test.

If an approximation of A obeying $\mathbf{c}$ changes $A(n)$ at stage s, then $U_{n,s}$ is listed as a component of the test. $A \models \mathbf{c}$ is used to show we indeed build a Solovay test.

Z is outside almost all components, so Z computes A correctly a.e.

Definition (ML-completeness for a cost function, GMNT 22)

Let $\mathbf{c} \geq \mathbf{c}_\Omega$ be a cost function. We say that a K-trivial A is smart for $\mathbf{c}$ if A is ML-complete among the sets that obey $\mathbf{c}$.

Thus $A \models \mathbf{c}$, and $B \leq_{ML} A$ for each $B \models \mathbf{c}$.

Theorem (GMNT 22, extending BGKNT 16 result for $\mathbf{c}_\Omega$)

For each $\mathbf{c} \geq \mathbf{c}_\Omega$ there is a c.e. set A that is smart for $\mathbf{c}$.

We may assume that $\mathbf{c}(k) \geq 2^{-k}$. Build A. There is a particular Turing functional Γ such that it suffices to show $A = \Gamma^Y \Rightarrow Y$ fails some $\mathbf{c}$-test.

- During construction, let
  $\mathcal{G}_{k,s} = \{Y : \Gamma_t^Y \upharpoonright 2^{k+1} \prec A_t \text{ for some } k \leq t \leq s\}$.
- Error set $\mathcal{E}_s$: those Y such that Γ_s^Y is to the left of A_s.
- Ensure $\lambda \mathcal{G}_{k,s} \leq \mathbf{c}(k,s) + \lambda(\mathcal{E}_s - \mathcal{E}_k)$. If this threatens to fail put next $x \in [2^k, 2^{k+1})$ into A. Then $\langle \mathcal{G}_k \rangle$ is the required $\mathbf{c}$-test.

ML-completeness for a cost function

Definition (recall)

Let $\mathbf{c} \geq \mathbf{c}_\Omega$ be a cost function. We say that a K-trivial A is smart for $\mathbf{c}$ if A is ML-complete among the sets that obey $\mathbf{c}$.

Theorem (GMNT 22)

For each K-trivial A there is a cost function $\mathbf{c}_A \geq \mathbf{c}_\Omega$ such that A is smart for $\mathbf{c}_A$.

This shows that there are no ML-minimal pairs, in a uniform way: if K-trivials A, B are noncomputable, there is a noncomputable c.e. D such that $D \models \mathbf{c}_A + \mathbf{c}_B$.
Then $D \leq_{ML} A, B$.

Smartness and half-bases

Recall:

Theorem (BGKNT 16)

Not every K-trivial is a half-base.

Proof (different from the original one).

- Ω_{even} and Ω_{odd} are low;
- If $Y \in \mathsf{MLR}$ is captured by a $\mathbf{c}_{\Omega}$-test, then it is superhigh.
- So a smart K-trivial is not a half-base.

□

A reducibility dual to $\leq_{ML}$

Definition

For $Z, Y \in \mathsf{MLR}$, by $Z \leq_{ML^*} Y$ we denote that for every K-trivial A,

$$A \leq_T Z \quad \Rightarrow \quad A \leq_T Y.$$

We say that $Z \in \mathsf{MLR}$ is feeble for $\mathbf{c}$ if Z is captured by a $\mathbf{c}$-test, and is of the least ML^*-degree among those. For example:

- For rational $p \in (0, 1)$, any appropriate "p-part" of Ω is feeble for $\mathbf{c}_{\Omega,p}$, where $\mathbf{c}_{\Omega,p}(x, s) = (\Omega_s - \Omega_x)^p$.
- Top degree: all ML-randoms captured by a $\mathbf{c}_\Omega$-test (i.e., the non Oberwolfach-randoms).
- Bottom degree: the weakly 2-randoms.

Pieces of Ω w.r.t. $\leq_{ML^*}$

- For any infinite computable $R \subseteq \mathbb{N}$, let Ω_R be the bits of Ω with position in R.
- can define a corresponding cost function $\mathbf{c}_{\Omega,R}$ similar to $\mathbf{c}_{\Omega,p}$:
$$A \text{ obeys } \mathbf{c}_{\Omega,R} \iff A \leq_T \Omega_R.$$
- Thus, Ω_R is feeble for $\mathbf{c}_{\Omega,R}$.

For each such R, let B_R be a K-trivial that is smart for $\mathbf{c}_{\Omega_R}$.

Theorem (GMNT 22)

The following are equivalent for infinite, computable $R, S \subseteq \mathbb{N}$:

1. $\Omega_S \leq_{ML^*} \Omega_R$;
2. $B_S \geq_{ML} B_R$;
3. There is $d \in \mathbb{N}$ such that $|S \cap n| \leq |R \cap n| + d$ for each n.

For instance, by (3), Ω_{even} and Ω_{odd} compute the same K-trivials!

Other weak reducibilities

- Note that $A \leq_T B$ if $J^A = \Psi^B$ for some functional Ψ (where $J^X = \phi_e^X(e)$ is the jump of X).
- Suppose B instead only can make a small number of guesses for $J^A(x)$?

Definition (N. 2009; related to Cole and Simpson 06)

We write $A \leq_{SJT} B$ if for each order function h, there is a uniform list $\langle \Psi_r \rangle$ of functionals such that $J^A(x)$, if defined, equals $\Psi_r^B(x)$ for some $r \leq h(x)$.

- A is strongly jump traceable (FNS 08) if $A \leq_{SJT} \emptyset$. These sets are properly contained in the K-trivials.
- There is no $\leq_{SJT}$-largest K-trivial, essentially by relativizing this.

Recall that Y is ω-c.a. if $Y \leq_{\mathrm{wtt}} \emptyset'$.
Let $\mathcal{C}$ be the class of the ω-c.a., superlow, or superhigh sets.

Theorem (Work without progress, with Greenberg and Turetsky)

The following are equivalent for K-trivial c.e. sets A, B.

(a) $A \leq_{SJT} B$

(b) $A \leq_T B \oplus Y$ for each $Y \in \mathcal{C} \cap \mathsf{MLR}$.

This generalises work of [GHN 2012] where $B = \emptyset$.
So we have on the K-trivials that

$$\leq_T \quad \Rightarrow \quad \leq_{ML} \quad \Rightarrow \quad \leq_{\omega-\text{c.a.}-ML}$$

$$\leq_T \quad \Rightarrow \quad \leq_{SJT} \quad \Rightarrow \quad \leq_{\omega-\text{c.a.}-ML}$$

and none of $\leq_{ML}$, $\leq_{SJT}$ implies the other.

Questions

- Is being a smart K-trival an arithmetical property?
- Stronger: is $\leq_{\mathrm{ML}}$ an arithmetical relation?
- Are the ML-degrees of the K-trivials dense?
- Can a smart K-trivial be cappable?
  Can it obey a cost function stronger than $\mathbf{c}_\Omega$?
- Is there an incomplete ω-c.a. ML-random above all the K-trivials?

Some references

- Bienvenu, Greenberg, Kučera, Nies, Turetsky: Coherent randomness tests and computing the K-trivial sets, JEMS 2016
- Greenberg, J. Miller, Nies: Computing from projections of random points, JML 2019
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