A Few Useful Facts

Properties of functions

- Exponentials
- Logarithms
- Summations
- Limits

Exponentials

$$
\begin{aligned}
a^{-1} &= 1/a \\
(a^m)^n &= a^{mn} \\
a^m a^n &= a^{m+n} \\
e^x &\geq 1 + x
\end{aligned}
$$

Any exponential function with base *a* > 1 grows faster than any polynomial

- **Lemma:** For any constants k>0 and a>1, $n^k = o(a^n)$.
- **Proof:** For any c>0, we need to find n_0 s.t. $n^k < c \cdot a^n$ for $n \geq n_0$.
  - $\frac{n}{\ln n}$ is an increasing function.
  - There exists n_0 such that $\frac{2k}{\ln a} < \frac{n}{\ln n}$ and $\frac{-2\log_a c}{\ln n} < \frac{n}{\ln n}$ for $n \geq n_0$.
  - We have $\frac{2k}{\ln a} + \frac{-2\log_a c}{\ln n} < \frac{2n}{\ln n} \rightarrow \frac{k}{\ln a} < \frac{n}{\ln n} + \frac{\log_a c}{\ln n}$
  - $k \log_a n = \frac{k \ln n}{\ln a} < n + \log_a c \rightarrow a^{k \log_a n} < c \cdot a^n \rightarrow n^k < c \cdot a^n$.
  - Hence, $n^k = o(a^n)$.

Logarithms

- **Binary log:** $\lg n = \log_2 n$
- **Natural log:** $\ln n = \log_e n$
- **Exponentiation:** $\lg^k n = (\lg n)^k$
- **Composition:** $\lg \lg n = \lg(\lg n)$

$$a = b^{\log_b a}$$

$$\log_c(ab) = \log_c a + \log_c b$$

$$\log_b a^n = n \log_b a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_b(1/a) = -\log_b a$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_b c} = c^{\log_b a}$$

Base of logarithm does not matter in asymptotics

$$\lg n = \Theta(\ln n) = \Theta(\log_{10} n)$$

Exponentials of different bases differ by an exponential factor

$$4^n = 2^n 2^n$$

Stirling's approximation

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

$$\log(n!) = \Theta(n \lg n)$$

Summations

Arithmetic Series

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$$

$$= \frac{1}{2} n(n+1) = \Theta(n^2)$$

Geometric series

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

$$= \frac{x^{n+1} - 1}{x - 1}$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} \text{ when } |x| < 1$$

Harmonic series

$$
\begin{aligned}
H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\
&= \sum_{k=1}^{n} \frac{1}{k} \\
&= \ln n + O(1)
\end{aligned}
$$

Limit

- Assume f(n), g(n)>0.

- $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0 \rightarrow f(n) = o(g(n))$
- $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) < \infty \rightarrow f(n) = O(g(n))$
- $0 < \lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) < \infty \rightarrow f(n) = \Theta(g(n))$
- $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) > 0 \rightarrow f(n) = \Omega(g(n))$
- $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = \infty \rightarrow f(n) = \omega(g(n))$

L'Hopital's Rule

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

Example

$$\lim_{n\to\infty} \frac{n\log n}{n^2}$$

$$= \lim_{n\to\infty} \frac{\log n}{n}$$

$$= \lim_{n\to\infty} \frac{1/n}{1}$$

L'Hopital's rule

$$= \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\implies n\log n \in o(n^2)$$