CS3230 Semester 1 2024/2025

Design and Analysis of Algorithms

# Tutorial 09
# Problem Reduction
# For Week 10

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## 1 Lecture Review: Problem Reduction

### 1.1 Revisiting Time Complexity

Time complexity is actually computed based on the input size.

- Example 1: Sorting
  Input: $N$ Integers.
  Input Size: $O(N \cdot \log N)$ for $N$ Integers, as each Integer requires $\log N$ bits.
  Merge sort algorithm runs in $O(N \log N)$ – polynomial w.r.t. input size
- Example 2: Fibonacci
  Input: One single Integer, which is $N$.
  Input Size: $O(\log N)$ for just that one Integer.
  DP algorithm (that sums the last two Fibonacci values) runs in $O(N)$ – this is NOT polynomial w.r.t. input size, as there is an exponential gap from $\log N$ to $N$, i.e., $2^{\log_2 N} = N$.
  But it is pseudopolynomial if we consider the input is $N$ and the DP runtime is $O(N)$.

### 1.2 Reductions

The Key Idea:

- We want to solve a problem A
- We know the solution to problem B
- To solve A, maybe we can translate/reduce problem A to B

Pseudo-code of a typical reduction algorithm:

```
Solve_A(instance_of_A):
    instance_of_B = translate_A_to_B(instance_of_A)
    solution_of_B = Solve_B(instance_of_B)
    solution_of_A = translate_B_to_A(solution_of_B)
    return solution_of_A
```

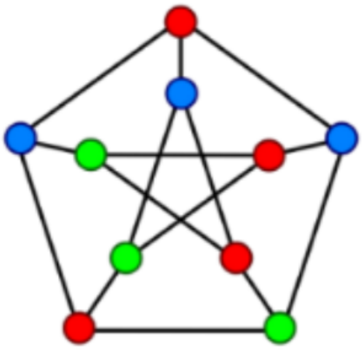
We call this **polynomial time reduction** if both sub-functions above (translate_A_to_B and translate_B_to_A) runs in polynomial time, and we denote this process as $A \leq_p B$.

### 1.3 Decision vs Optimization Problems

- Decision Problem: Any problem where the output is Boolean (YES/NO)
- Optimization Problem: Any problem where we want to optimize the output
  Synonyms of optimize are: maximize, minimize, most optimal, longest, shortest, etc.

## 2 Tutorial 09 Questions

Q1). Graph-Coloring



Which statement(s) is/are True?

1. If we can solve the optimization problem for Graph-Coloring in polynomial time, we would be able to solve the decision problem for Graph-Coloring in polynomial time.
2. If we can solve the decision problem for Graph-Coloring in polynomial time, we would be able to solve the optimization problem for Graph-Coloring in polynomial time.
3. If the decision problem for Graph-Coloring cannot be solved in polynomial time, the optimization problem for Graph-Coloring cannot be solved in polynomial time.
4. If the optimization problem for Graph-Coloring cannot be solved in polynomial time, the decision problem for Graph-Coloring cannot be solved in polynomial time.

Q2). Partition versus Ball-Partition

- PARTITION: Given a set of positive integer $S$, can the set be partitioned into two sets of equal total sum? For example, if $S = \{18, 2, 8, 5, 7, 24\}$, we can partition $S$ into $S_1 = \{18, 2, 5, 7\}$ and $S_2 = \{8, 24\}$, both sum to 32.
- BALL-PARTITION: Given $k$ balls, can we divide the balls into two boxes with an equal number of balls? For example, if $k = 4$, we can divide the balls into $\{2, 2\}$.

We try to show that PARTITION $\leq_p$ BALL-PARTITION using the following transformation $A$:

1. From the problem PARTITION, we are given a set of positive integers $S$.
2. We define $k$ as the total sum of all integers in $S$.
3. We use this number $k$ for the BALL-PARTITION problem.

What is wrong with this transformation?

1. The transformation does not run in polynomial time.
2. This transformation is correct.
3. A YES solution to $A(S)$ does not imply a YES solution to $S$.
4. A YES solution to $S$ does not imply a YES solution to $A(S)$.

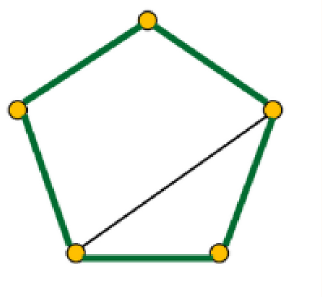
Q3). PARTITION versus KNAPSACK (as in Lecture)

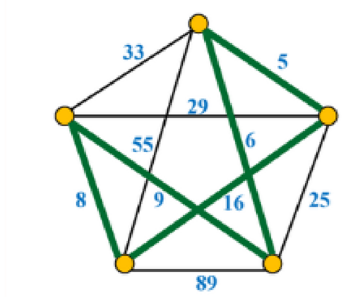
Transformation: Given a PARTITION instance $\{w_1, w_2, \ldots, w_n\}$ with total sum $S = \sum_{i=1}^{n} w_i$, construct a KNAPSACK instance $\{(w_1, w_1), (w_2, w_2), \ldots, (w_n, w_n)\}$ with capacity $W = \frac{S}{2}$ and threshold $V = \frac{S}{2}$.

Which statement(s) is/are True?

1. The transformation runs in polynomial time.
2. A YES answer to the PARTITION instance implies a YES answer to the KNAPSACK instance.
3. A YES answer to the KNAPSACK instance implies a YES answer to the PARTITION instance.

Q4). HAMILTONIAN-CYCLE (HC) versus TRAVELLING-SALESPERSON-PROBLEM (TSP) (as in Lecture)






Show that HC $\leq_p$ TSP.

Show that HC $\leq_p$ TSP!

The steps:

1. Show the transformation algorithm.
2. Show the transformation algorithm runs in polynomial time.
3. Show A YES answer to the HC instance implies a YES answer to the TSP instance.
4. Show A YES answer to the TSP instance implies a YES answer to the HC instance.