NATIONAL UNIVERSITY OF SINGAPORE

CS1231 - DISCRETE STRUCTURES

(Semester 1: AY2015/16)

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your Student Number (Matriculation Number) only. **Do not write your name.**
2. This assessment paper contains **TWO (2)** parts and comprises **FIFTEEN (15)** printed pages, including this page.
3. Answer **ALL** questions.
4. This is an **OPEN BOOK** assessment.
5. You are allowed to use **NUS APPROVED CALCULATORS**.
6. You may use pen or pencil, but please erase cleanly, and write legibly.
7. Please write your Student Number below.

STUDENT NUMBER: ___________________________________

EXAMINERS' USE ONLY			
Part	MaxScore	Mark	Remark
A	30	OCR	OCR
Q16	12		
Q17	14		
Q18	12		
Q19	12		
Subtotal Q16-Q19	50		

Part A: (30 marks) MCQ. Answer on the OCR form.

For each multiple choice question, choose the best answer and **shade** the corresponding choice on the OCR form. Each multiple choice question is worth 2 marks. No mark is deducted for wrong answers. Shade your student number (check that it is correct!) on the OCR form as well. You should use a **2B pencil**.

Q1. Find $x \in \mathbb{Z}$ such that $4x \equiv 2 \pmod 6$.

- A. 1/2
- B. 2
- C. 3
- D. 1000
- E. There is no solution.

Q2. Consider the relation $\equiv_{(\text{mod } 6)}$ defined on $\mathbb{Z}$. If $x^2 \in [27]$, then x could be

- A. 7
- B. 8
- C. 9
- D. 10
- E. 11

Q3. Aaron wants to set multiple choice questions for the CS1231 examination. He intends to give every student the same questions, but have each student see the questions in a different sequence. If there are 333 students in the class, what is the least number of questions Aaron must set?

- A. 5
- B. 6
- C. 9
- D. 12
- E. 333

Q4. Which of the following statements is **false**?

- A. $\sim(p \to \sim q) \equiv p \wedge q$
- B. $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$
- C. $(p \to q) \to r \equiv (p \wedge \sim q) \vee r$
- D. $(p \to q) \to r \equiv p \to (q \to r)$
- E. None of the above.

Q5. On the island of knights (who always tell the truth) and knaves (who always lie), you meet two natives A and B. A says: "I am a knave or B is a knight". What are A and B?

- A. Both A and B are knights.
- B. A is a knave and B is a knight.
- C. A is a knight and B is a knave.
- D. Both A and B are knaves.
- E. Cannot be determined.

Q6. How many spanning trees does the complete graph K_4 have?

A. 4

B. 8

C. 15

D. 16

E. 20

Q7. Five friends go to a restaurant for dinner. The restaurant offers four types of main courses, and each person will have one. What is the number of possible orders the chef could get to see? (Example of an order: "Two fish-n-chips, one portobello steak, and two carbonara pastas".)

A. 20

B. 56

C. 120

D. 126

E. None of the above.

Q8. Suppose one urn contains 5 blue balls and 5 gray balls, and a second urn contains 3 blue balls and 7 gray balls. Both urns are equally likely to be chosen. You draw a ball at random from one of the two urns. If a blue ball is drawn, what is the probability that it comes from the first urn?

A. 5/8

B. 2/5

C. 2/3

D. 3/5

E. 1/2

Q9 to Q13 refer to the following definitions. Answer each question independently.

Define functions f, g and h as follows:

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad \forall x \in \mathbb{R},\ f(x) = x^2.$$
$$g : \mathbb{N} \longrightarrow \mathbb{N}, \quad \forall x \in \mathbb{N},\ g(x) = x^2.$$
$$h : A \longrightarrow B, \quad \forall x \in A,\ h(x) = x^2.$$
$$\text{where } A = \{0, 1, 2, 3, 4\},\ \text{ and } B = \{0, 1, 4, 9, 16\}.$$

Q9. Which function is one-to-one?

A. f only.
B. g only.
C. h only.
D. g and h only.
E. All of the functions.

Q10. Which function is onto?

A. f only.
B. g only.
C. h only.
D. g and h only.
E. f and h only.

Q11. Which function has a preimage of 3?

A. f only.
B. g only.
C. f and g only.
D. g and h only.
E. All of the functions.

Q12. Let $P : C \longrightarrow D$ be a function, and let $U \subseteq C$. We denote P_U to mean the restriction of P to U. That is, $P_U : U \longrightarrow D, \quad \forall x \in U,\ P_U(x) = P(x)$.

Which of the following statements is true?

A. $\exists U \subseteq \mathbb{R}$ such that $f_U = g$
B. $\exists U \subseteq \mathbb{N}$ such that $g_U = f$
C. $\exists U \subseteq A$ such that $h_U = g$
D. $\exists U \subseteq A$ such that $h_U = f$
E. None of the above.

Q13. Which of the following statements is true?

A. $f \circ f$ is onto.

B. $h \circ h$ is a relation.

C. g^{-1} is a function.

D. $g \circ g$ is one-to-one.

E. None of the above.

Q14 to Q15 refer to the following definitions. Answer each question independently.

> Let $A = \{1, 2, 3, 4\}$. Since each element of $\mathcal{P}(A \times A)$ is a subset of $A \times A$, it is a binary relation on A.

Q14. Assuming each relation in $\mathcal{P}(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is *reflexive*?

A. $1/2^2$

B. $1/2^4$

C. $1/2^6$

D. $1/2^{12}$

E. $1/2^{16}$

Q15. Assuming each relation in $\mathcal{P}(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is *symmetric*?

A. $1/2^2$

B. $1/2^4$

C. $1/2^6$

D. $1/2^{12}$

E. $1/2^{16}$

Part B: (50 marks) Structured questions. Write your answer in the space provided. Marks may be deducted for unnecessary statements in proofs.

Q16. (12 marks)

(a) Define $|x|$ to be the absolute value of an integer x, that is, if x is non-negative, $|x| = x$, else $|x| = -x$.

Given the set $S = \{-9, -6, -1, 3, 5, 8\}$, for each of the following statements state whether it is true or false, with explanation.

i. (2 marks) $\exists z \in S$ such that $\forall x, y \in S,\ z > |x - y|$.

ii. (2 marks) $\exists z \in S$ such that $\forall x, y \in S,\ z < |x - y|$.

Suppose we now treat S as a general finite set with $k > 1$ elements listed from smallest to largest.

iii. (2 marks) Describe a strategy to find the desired z in part i.

iv. (2 marks) Describe a strategy to find the desired z in part ii.

(This page is intentionally left blank. You may write in it.)

(b) (4 marks) Consider the equation:

$$3x^2 - 4xy + y^2 = 0.$$

For each statement below, determine whether it is true or false. If the statement is true, prove it; otherwise, disprove it.

i. For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that the equation is true.

ii. There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, the equation is true.

Q17. (14 marks)

(a) (4 marks) You wish to select five persons from seven men and six women to form a committee that includes at least three men.

i. In how many ways can you form the committee?

ii. If you randomly choose five persons to form the committee, what is the probability that you will get a committee with at least three men?

(b) (5 marks) In an orientation game with girls and boys, chairs are placed in a row and pupils are to sit on them, one person on each chair. The rule, however, states that no two boys are allowed to sit next to each other.

Let n, a positive integer, denote the number of chairs, and let $f(n)$ be the number of ways the chairs can be filled. For example, let G denote a girl, and B a boy. Then for $n = 3$, the chairs may be filled with GGG, GGB, GBG, BGG, or BGB, for a total of 5 ways (ie. $f(3) = 5$).

i. Write a recurrence relation for $f(n)$, including initial conditions.

ii. How many ways are there to fill ten chairs?

(c) (5 marks) Prove that among any five points selected inside a square with a side length of 2 units, there always exists a pair of these points that are within $\sqrt{2}$ units of each other.

(*Hint:* Apply the Pigeonhole Principle.)

Q18. (12 marks)

(a) (4 marks) Draw a simple graph with six vertices and ten edges, and where each vertex has a degree of two or four.

(b) (4 marks) Let $E(K_n)$ denote the number of edges in a complete graph K_n of n vertices. Write an explicit formula for $E(K_n)$ and prove it.

(c) (4 marks) Prove the following:
A connected graph G has an Euler path if, and only if, exactly two vertices have odd degree.

Q19. (12 marks)

(a) An integer sequence u_n is defined recursively as:

$$u_{n+1} = \begin{cases} 1, & \text{if } n = 0, \\ (n+1)^2\, u_n, & \text{if } n > 0. \end{cases}$$

i. (3 marks) Determine u_2, u_3 and u_4.

ii. (3 marks) Prove, by Mathematical Induction, that $u_n = (n!)^2$, for $n \in \mathbb{Z}^+$.

(b) Another integer sequence, s_n, is defined recursively as:

$$s_n = \begin{cases} 0, & \text{if } n = 0, \\ s_{n-1} + (-1)^n\, n^2, & \text{if } n > 0. \end{cases}$$

i. (3 marks) Determine s_1, s_2, s_3 and s_4.

ii. (3 marks) Find an explicit formula for s_n. (You do not need to prove the formula using Mathematical Induction; instead, just explain how you obtain it.)

END OF PAPER