8. Relations (Part 1)

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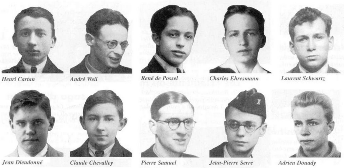
8.1. Introduction

> *For whereas in the past it was thought that every branch of mathematics depended on its own particular intuition which provided its concepts and prime truths, nowadays it is known to be possible, logically speaking, to derive practically the whole of known mathematics from a single source, the Theory of Sets.*
>
> Nicolas Bourbaki



Reading

Section 1.2, 1.3, 8.1 of Epp.

Motivation

Suppose you work for Facebook. It is well-known that Facebook derives a major part of its income from advertising.

That is why a portion of a Facebook page is devoted to ads.

Marketing at Bulwark Exterminating
Studied at UOP
Lives in Queen Creek, Arizona
Married

Thomas Ballantyne shared a link via NightRider Jewelry.
20 minutes ago

My little Brothers... All grown up and making headlines. Way to go Night Rider.

Tempe jeweler expanding
www.azcentral.com
Tempe jeweler expanding, In addition to designer, artist and business owner, Nick Ballantyne can add magazine-cover man to his resume.

Thomas likes NightRider Jewelry.
Thomas and Loree Olsen are now friends.

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Would You Vote for Mitt Romney in 2012? Voice Your Opinion in Today's National Survey.

Motivation

Mike, a famous sportswear company, wants to advertise in Facebook, but wants to show different ads to different users depending on their profile information.

Specifically, by looking at an FB user's gender, birthdate, occupation, sporting interests, and pages that the user liked, Mike has defined 10 different target groups of customers.

MIKE

Example: TG1 is for young females who work in IT companies, and who love running. TG2 is for older male teachers who lead a sedantary lifestyle.

Motivation

Thus, your job is to partition all FB users into these 10 target groups, and give Mike a sample of 50 users in each of these target groups for Mike to test their advertising campaign.

But what does it mean to partition? How are users in one target group related to each other, or to those in a different target group?

MIKE

Mike also wants to rank all target groups according to their “willingness to buy sporting products”, which is measurable from the users’ profile info. What does it mean to rank different users? Can a ranking even be possible?

The topic of Relations will help answer these questions.

Definition 8.1.1 (Ordered Pair)

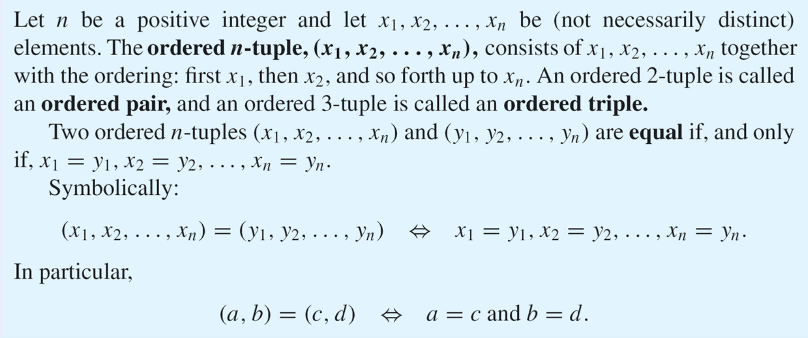
Let S be a non-empty set, and let x, y be two elements in S. The ordered pair, denoted (x, y), is a mathematical object in which the first element of the pair is x and the second element is y.

Two ordered pairs (x, y) and (a, b) are equal iff $x = a$ and $y = b$.

Examples:

- On a 2D map using Cartesian coordinates, one specifies a point on the map as (x, y). Typically, this means the point is x units in the X-direction and y units in the Y-direction, relative to an Origin.
- $(3, 4) \neq (4, 3)$.

Definition 8.1.2 (Ordered n-tuple)



Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Examples:

- $(1, 4, 3) \neq (1, 3, 4)$, because the order matters.
- $((1, 2), 3) \neq (1, 2, 3)$, because one object is an ordered pair, the other is an ordered triple.

Definition 8.1.3

Let S and T be two sets. The Cartesian product (or cross product) of S and T, noted $S \times T$ is the set such that:

$$\forall X\ \forall Y\ (\ (X, Y) \in S \times T \leftrightarrow (X \in S) \wedge (Y \in T))$$

Notice that the Cartesian product is neither commutative nor associative.

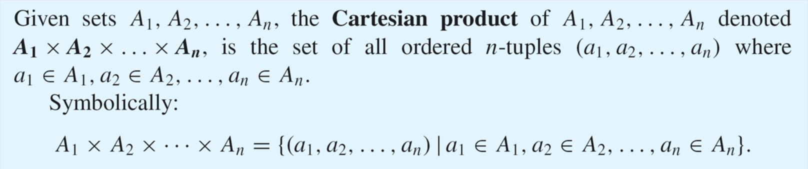
Example of Cartesian Product

This is a tabular representation of the Cartesian product of $S = \{1, 2, 3\}$ and $T = \{a, b\}$.

$$S \times T = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

1	a
1	b
2	a
2	b
3	a
3	b

Definition 8.1.4 (Generalized Cartesian Product)



Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\boldsymbol{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

If V is a set of sets, then the Generalized Cartesian product of its elements will be written as:

$$\prod_{S \in V} S$$

8.2. Relations

Definition 8.2.1

Let S and T be two sets. A binary relation from S to T, noted $\mathcal{R}$, is a subset of the Cartesian product $S \times T$.

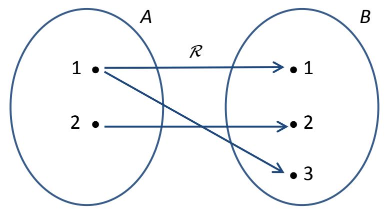
$s \mathcal{R} t$ stands for $(s, t) \in \mathcal{R}$.
$x \not\mathcal{R} y$ stands for $(x, y) \notin \mathcal{R}$.

Example 1

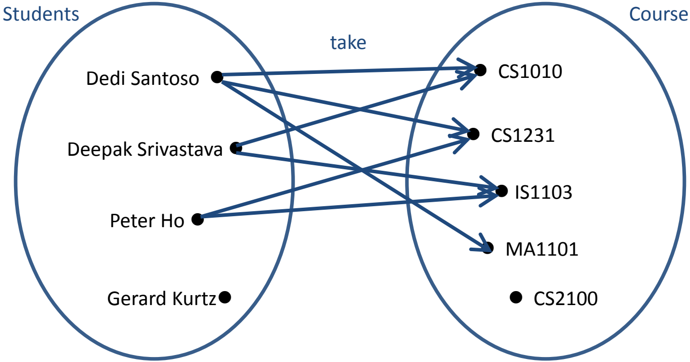
Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, and define a relation $\mathcal{R}$ from A to B as follows:

$$\forall a \in A, \forall b \in B \ (\ (a, b) \in \mathcal{R} \quad \leftrightarrow \quad (a - b) \text{ is even})$$

It is easy to see that $\mathcal{R} = \{(1, 1), (1, 3), (2, 2)\}$.



Example 2: Diagram showing students taking courses as a relation.



Let $\mathcal{R} \subseteq S \times T$ be a binary relation from S to T.

Definition 8.2.2

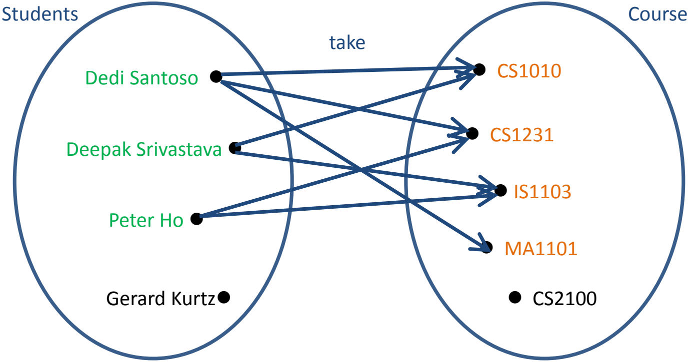
The domain of $\mathcal{R}$ is the set $\mathcal{D}om(\mathcal{R}) = \{s \in S \mid \exists t \in T\ (s\ \mathcal{R}\ t)\}$.

Definition 8.2.3

The image (or the range) of $\mathcal{R}$ is the set
$\mathcal{I}m(\mathcal{R}) = \{t \in T \mid \exists s \in S\ (s\ \mathcal{R}\ t)\}$.

Definition 8.2.4

The co-domain of $\mathcal{R}$ is the set $co\mathcal{D}om(\mathcal{R}) = T$.



$\mathcal{D}om(\text{take}) = \{$ Dedi Santoso, Deepak Srivastava, Peter Ho$\}$.
$\mathcal{I}m(\text{take}) = \{$ CS1010, CS1231, IS1103, MA1101 $\}$.
$co\mathcal{D}om(\text{take}) =$ Course.

Proposition 8.2.5

Let $\mathcal{R}$ be a binary relation. Then $\mathcal{I}m(\mathcal{R}) \subseteq co\mathcal{D}om(\mathcal{R})$.

Proof omitted.

Definition 8.2.6

Let S and T be sets. Let $\mathcal{R} \subseteq S \times T$ be a binary relation. The inverse of the relation $\mathcal{R}$, denoted $\mathcal{R}^{-1}$, is the relation from T to S such that:

$$\forall s \in S, \forall t \in T \ (t \ \mathcal{R}^{-1} \ s \ \leftrightarrow \ s \ \mathcal{R} \ t)$$

The inverse Student-Course relation.