6. Sets

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6.1. Introduction

A set is a Many that allows itself to be thought of as a One.

Georg Cantor

Reading

Section 6.1 — 6.3 of Epp.
Section 3.1 — 3.4 of Campbell.

Familiar concepts

- Sets can be defined in extension, by explicitly listing its members: $\{1, 2, 3\}$, $\{apple, orange, red, unicorn\}$.
- Membership: $1 \in \{1, \{1, 2\}\}$.
- non-membership: $3 \notin \{1, 2\}$.
- No duplicates: $\{1, 1, 2, 2, 2\} = \{1, 2\}$
- Order does not matter: $\{1, 2\} = \{2, 1\}$.

A set can also be defined in intention, by specifying a property that characterizes its members:

$$\{X \in \mathbb{N} \mid X < 1 \vee X > 5\}.$$

Definition 6.1.1

S is a subset of T (or S is contained in T, or T contains S, or T is a superset of S) if all the elements of S are elements of T. We write $S \subseteq T$.[1] Mathematically,

$$S \subseteq T \leftrightarrow \forall x \in S, x \in T.$$

Examples:

- $\{1, 2\} \subseteq \{1, 2, 3\}$
- $\{1, 2, 3\} \subseteq \{1, 2, 3\}$. A set is a subset of itself.
- $\{3, 4\} \nsubseteq \{1, 2, 3\}$

[1] Some authors use $X \subset Y$.

Warning

Do not confuse $S \in T$ with $S \subseteq T$!

Example: Let $S = \{1, 2, \{3, 4\}\}$, then

- $\{3, 4\} \in S$, but $\{3, 4\} \not\subseteq S$.
- $\{4\} \not\subseteq S$, and $4 \notin S$.

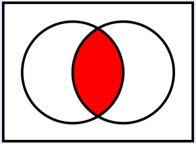
Note that the definition of subset allows a set to be a subset of itself. Sometimes we speak of a proper subset:

A set S is a proper subset, of T, denoted $S \subsetneq T$, iff $S \subseteq T$ and there is at least one element in T that is not in S.

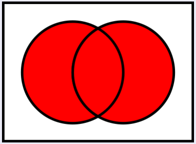
Example: $\{1, 2\} \subsetneq \{1, 2, 3\}$.

Therefore, a set cannot be a proper subset of itself.

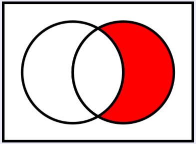
Venn diagrams and set operations



Intersection: $A \cap B$



Union: $A \cup B$



Difference: $B \backslash A$ or $B - A$

6.2. Basic Set Theory

The formal set theory is called Zermelo-Fraenkel Set Theory with the Axiom of Choice, or ZFC. But this is outside the scope of CS1231. Instead, we will study basic set theory.

It is customary to let $\mathcal{U}$ denote the universal set, or the universe of discourse, that contains all the objects in our context of discussion.

Definition 6.2.1 (Empty set)

An empty set has no element, and is denoted as $\varnothing$ or $\{\}$.

Mathematically, $\varnothing$ is such that:

$$\forall Y \in \mathcal{U},\ Y \notin \varnothing. \tag{1}$$

Theorem 6.2.4 (Epp) An Empty Set is a Subset of all Sets

For all sets $S, \varnothing \subseteq S$

Proof: see page 324 of Epp.

Definition 6.2.2 (Set Equality)

Two sets are equal if and only if they have the same elements.

Examples:

- $\{1, 2, 3\} = \{2, 1, 3, 2\}$.
- $\{\} \neq \{\{\}\}$.

Proposition 6.2.3

For any two sets X and Y, X is a subset of Y and Y is a subset of X if, and only if, $X = Y$.

$$\forall X\ \forall Y\ ((X \subseteq Y \wedge Y \subseteq X) \leftrightarrow X = Y)$$

Proof omitted. You try!

Note that this gives us a way to check if two sets are equal: by checking if one is a subset of the other, and vice versa.

Corollary 6.2.5 (Epp) The Empty Set is Unique

Proof:

1. Let X_1 and X_2 be two empty sets.
2. Therefore $X_1 \subseteq X_2$ by Theorem 6.2.4 (Epp).
3. Therefore $X_2 \subseteq X_1$ by Theorem 6.2.4 (Epp).
4. Therefore $X_1 = X_2$ by Proposition 6.2.3. ■

Definition 6.2.4 (Power Set)

Given any set S, the power set of S, denoted by $\mathcal{P}(S)$, or 2^S, is the set whose elements are all the subsets of S.

That is, given set S, if $T = \mathcal{P}(S)$, then:

$$T = \{ X \mid X \subseteq S \}.$$

Examples:

- $\mathcal{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$.
- $\mathcal{P}(\emptyset) = \{\emptyset\}$.

If S has n elements, then 2^S has 2^n elements.

6.3. Operation on Sets

Definition 6.3.1 (Union)

Let S be a set of sets, then we say that T is the union of the sets in S, and write:

$$T = \bigcup S = \bigcup_{X \in S} X$$

iff each element of T belongs to *some* set in S.
That is, given S, the set T is such that:

$$T = \{\ y \in \mathcal{U} \mid y \in X \text{ for some } X \in S\ \}.$$

For two sets A, B, we may simply write $T = A \cup B$.

Examples:

- Let $\{1,2\} \cup \{3,1\} = \{1,2,3\}$.
- Let $S = \{\{1,2\},\{3\},\{1,\{2\}\}\}$. Then $T = \{1,2,3,\{2\}\}$.

Proposition 6.3.2 (Some easy propositions)

Let A, B, C be sets. Then,

- $\bigcup \varnothing = \bigcup_{A \in \varnothing} A = \varnothing$
- $\bigcup \{A\} = A$
- $A \cup \varnothing = A$
- $A \cup B = B \cup A$
- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cup A = A$
- $A \subseteq B \leftrightarrow A \cup B = B$

All are easy to prove. You try!

Definition 6.3.3 (Intersection)

Let S be a non-empty set of sets. The intersection of the sets in S is the set T whose elements belong to *all* the sets in S.

That is, given S, the set T is such that:

$$T = \{\ y \in \mathcal{U} \mid \forall X\ ((X \in S) \to (y \in X))\ \}.$$

We write:

$$T = \bigcap S = \bigcap_{X \in S} X$$

For two sets A, B, we may simply write $T = A \cap B$.

Examples:

- $\{1, 2, 3\} \cap \{1, 4, 2, 5\} = \{1, 2\}$.
- Let $S = \{\{1, 2\}, \{3\}, \{\{1\}, \{2\}\}\}$. Then $T = \varnothing$.

Proposition 6.3.4 (Some easy propositions)

Let A, B, C be sets. Then,

- $A \cap \varnothing = \varnothing$
- $A \cap B = B \cap A$
- $A \cap (B \cap C) = (A \cap B) \cap C$
- $A \subseteq B \leftrightarrow A \cap B = A$

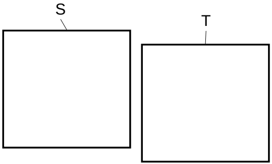
Distributivity laws:

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proofs omitted. You try!

Definition 6.3.5 (Disjoint)

Let S and T be two sets. S and T are disjoint iff $S \cap T = \varnothing$.



Definition 6.3.6 (Mutually disjoint)

Let V be a set of sets. The sets $T \in V$ are mutually disjoint iff every two distinct sets are disjoint.

$$\forall X, Y \in V \ (X \neq Y \rightarrow X \cap Y = \varnothing).$$

Example:
The sets in $V = \{\{1,2\},\{3\},\{\{1\},\{2\}\}\}$ are mutually disjoint.

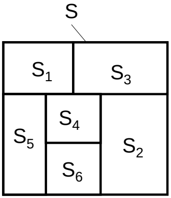
Definition 6.3.7 (Partition)

Let S be a set, and let V be a set of non-empty subsets of S. Then V is called a partition of S iff

(i) The sets in V are mutually disjoint.

(ii) The union of the sets in V equals S.

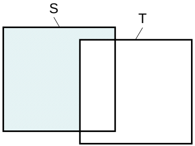
The Venn diagram shows $\{S_1, \ldots, S_6\}$ is a partition of S.



Definition 6.3.8 (Non-symmetric Difference)

Let S and T be two sets. The (non-symmetric) difference (or relative complement) of S and T, denoted $S - T$[2] is the set whose elements belong to S and do not belong to T.

$$S - T = \{\, y \in \mathcal{U} \mid y \in S \wedge y \notin T \,\}.$$



Examples:

- $\{1, 2, 3, 5, 8\} - \{1, 2, 4, 8, 16, 32\} = \{3, 5\}$.
- $\{1, 2, 4, 8, 16, 32\} - \{1, 2, 3, 5, 8\} = \{4, 16, 32\}$.

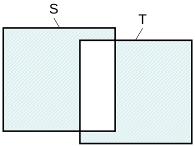
[2] Some authors use $S \setminus T$.

Definition 6.3.9 (Symmetric Difference)

Let S and T be two sets. The symmetric difference of S and T, denoted $S \ominus T$[3] is the set whose elements belong to S or T but not both.

$$S \ominus T = \{\ y \in \mathcal{U} \mid y \in S \oplus y \in T\ \}.$$

Recall that $\oplus$ means "exclusive-or".

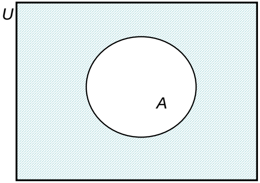


Example:
$\{1, 2, 3, 5, 8\} \ominus \{1, 2, 4, 8, 16, 32\} = \{3, 4, 5, 16, 32\}.$

[3] Some authors use $S \Delta T$.

Definition 6.3.10 (Set Complement)

Let $A \subseteq \mathcal{U}$. Then, the complement (or absolute complement) of A, denoted A^c, is $\mathcal{U} - A$.



Example:
$\mathcal{U} = \mathbb{N}$, and $A = \{\text{positive even numbers}\}$. Then $A^c = \{\text{positive odd numbers}\} \cup \{0\}$.

There are many useful identities and theorems in set theory, many of which are already familiar to you.

Please read Theorems 6.2.1 to 6.2.3 (Epp), and their proofs. You may use and cite them as needed. Example:

$$\text{For all sets } A, B : (A \cup B)^c = A^c \cap B^c.$$

This is De Morgan's law on sets. Let's prove this.

Proof:

1. Take any two sets: A, B.
2. (Need to show that $(A \cup B)^c \subseteq A^c \cap B^c$)
3. For any $x \in (A \cup B)^c$:
4. $x \notin (A \cup B)$, by definition of complement.
5. So $\sim(x \in A \vee x \in B)$, by definition of union.
6. Thus $x \notin A \wedge x \notin B$, by De Morgan's laws.
7. Thus $x \in A^c \wedge x \in B^c$, by definition of complement.
8. Thus $x \in A^c \cap B^c$, by definition of intersection.
9. Thus $(A \cup B)^c \subseteq A^c \cap B^c$, by definition of subset.

...

Proof cont'd:

10. (Now, need to show that $A^c \cap B^c \subseteq (A \cup B)^c$)
11. For any $x \in A^c \cap B^c$:
12. $x \in A^c \wedge x \in B^c$, by definition of intersection.
13. Thus $x \notin A \wedge x \notin B$, by definition of complement.
14. Thus $\sim(x \in A \vee x \in B)$, by De Morgan's laws.
15. Thus $x \notin A \cup B$, by definition of union.
16. Thus $x \in (A \cup B)^c$, by definition of complement.
17. Thus $A^c \cap B^c \subseteq (A \cup B)^c$, by definition of subset.
18. Hence $(A \cup B)^c = A^c \cap B^c$, by Proposition 6.2.3. ■

Summary

- Sets may be defined in extension or in intention.
- Set membership, subset, and set equality are basic properties.
- Operations on sets include: union, intersection, difference, complement.
- Set identities mirror those of logic equivalences.
  - Set complement is like logical negation.
  - Set union is like logical or.
  - Set intersection is like logical and.
  - De Morgan's laws apply to sets too.