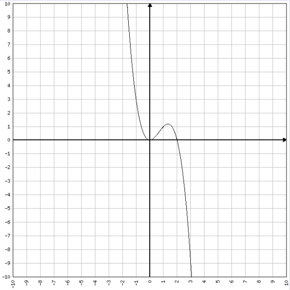
7. Functions

Terence Sim

7.1. Functions

"$f(x) = x^2 \times (b - x) = d$."
Treatise on Algebra
Sharaf al-Din al-Tusi



Reading

Chapter 3.7, 3.8 of Campbell.
Chapter 1.3, 7.1 — 7.3 of Epp.

Definition 7.1.1

Let f be a relation such that $f \subseteq S \times T$. Then f is a function from S to T, denoted $f : S \to T$ iff

$$\forall x \in S,\ \exists y \in T\ (x\ f\ y \land (\forall z \in T\ (x\ f\ z \to y = z)))$$

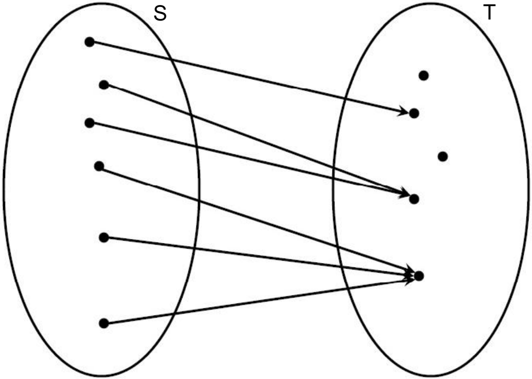
Notation for uniqueness

$$\exists! x \in T \; p(x) \;\equiv\; \exists x \in T \; (p(x) \wedge \forall y \in T \; (p(y) \to x = y)).$$

Definition 7.1.1 becomes:
Let f be a relation such that $f \subseteq S \times T$. Then f is a function from S to T, denoted $f : S \to T$ iff

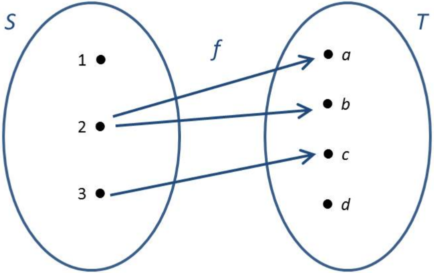
$$\forall x \in S, \; \exists! y \in T \; (x \; f \; y).$$

Let $f : S \to T$ be a function. We write $f(x) = y$ (or $x \mapsto y$) iff $(x, y) \in f$. The relation notation is $x \; f \; y$. Also, x is called the argument of f.



Every dot in S must have exactly one outgoing arrow.

Is this a function?



Definition 7.1.2

Let $f : S \to T$ be a function. Let $x \in S$. Let $y \in T$ such that $f(x) = y$. Then x is called a pre-image of y.

Definition 7.1.3

Let $f : S \to T$ be a function. Let $y \in T$. The inverse image of y is the set of all its pre-images: $\{x \in S \mid f(x) = y\}$.

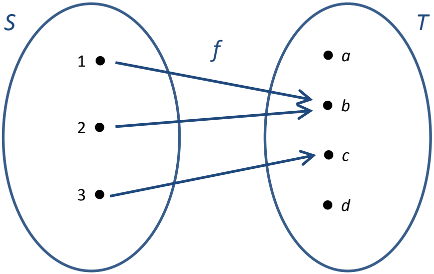
Definition 7.1.4

Let $f : S \to T$ be a function. Let $U \subseteq T$. The inverse image of U is the set that contains all the pre-images of all elements of U:
$\{x \in S \mid \exists y \in U,\ f(x) = y\}$.

Definition 7.1.5

Let $f : S \to T$ be a function. Let $U \subseteq S$. The restriction of f to U is the set: $\{(x, y) \in U \times T \mid f(x) = y\}$.

Example:



- The pre-image of c is 3.
- The inverse image of b is $\{1, 2\}$.
- The inverse image of $\{a, d\}$ is $\varnothing$.
- The inverse image of T is $\{1, 2, 3\}$.
- The restriction of f to $\{2, 3\}$ is $\{(2, b), (3, c)\}$.

7.2.1. Injective

Definition 7.2.1

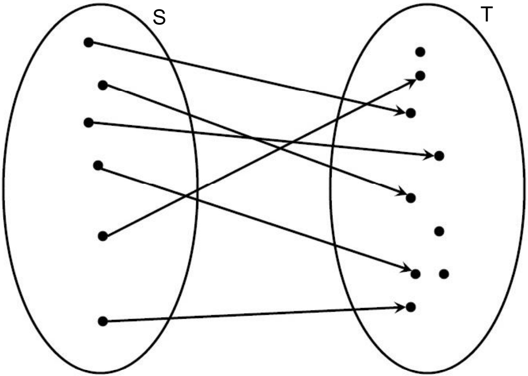
Let $f : S \to T$ be a function. f is injective iff

$$\forall y \in T, \forall x_1, x_2 \in S \ ((f(x_1) = y \land f(x_2) = y) \to x_1 = x_2).$$

We also say that f is an injection or that f is one-to-one.

Examples:
The function $f : \mathbb{Z} \to \mathbb{Z}$, where $\forall x \in \mathbb{Z},\ f(x) = x + 1$ is injective.

But the function $g : \mathbb{Z} \to \mathbb{Z}$, where $\forall x \in \mathbb{Z},\ g(x) = x^2$ is not injective.



Every dot in T has at most one incoming arrow.

7.2.2. Surjective

Definition 7.2.2

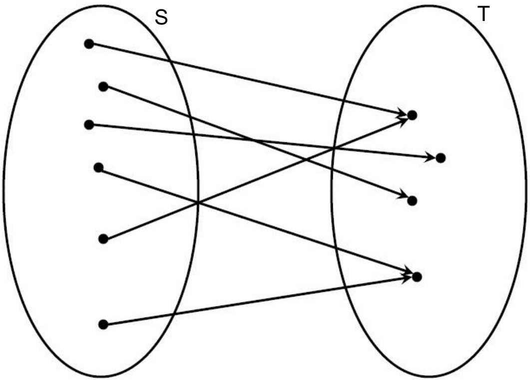
Let $f : S \to T$ be a function. f is surjective iff

$$\forall y \in T,\ \exists x \in S\ (f(x) = y).$$

We also say that f is a surjection or that f is onto.

Examples:
The function $f : \mathbb{Z} \to \mathbb{Z}$, where $\forall x \in \mathbb{Z},\ f(x) = x + 1$ is surjective.

But the function $g : \mathbb{Z} \to \mathbb{Z}$, where $\forall x \in \mathbb{Z},\ g(x) = x^2$ is not surjective.

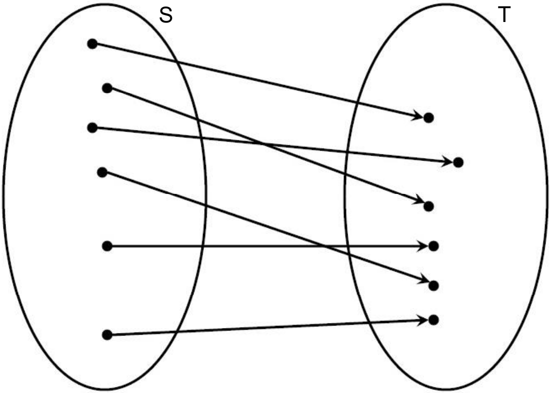


Every dot in T has at least one incoming arrow.

7.2.3. Bijective

Definition 7.2.3

Let $f : S \to T$ be a function. f is bijective iff f is injective and f is surjective. We also say that f is a bijection.



Every dot in T has exactly one incoming arrow.

7.2.4. Inverse

Proposition 7.2.4

Let $f : S \to T$ be a function and let f^{-1} be the inverse relation of f from T to S. Then f is bijective iff f^{-1} is a function.

This Proposition tells us when the inverse of a function exists.

Proof: forward direction

(Need to prove: if f is bijective then f^{-1} is a function)

1. Assume that f is bijective:
2. Then f is surjective by definition of bijective.
3. Thus $\forall y \in T,\ \exists x\ (x\ f\ y)$ by definition of surjective.
4. Thus $\forall y \in T,\ \exists x\ (y\ f^{-1}\ x)$ by definition of the inverse relation.
5. Also, f is injective by definition of bijective.
6. Thus $\forall y \in T,\ \forall x_1 \in S,\ \forall x_2 \in S\ ((x_1\ f\ y \wedge x_2\ f\ y) \rightarrow x_1 = x_2)$ by definition of injective.
7. So $\forall y \in T, \forall x_1 \in S, \forall x_2 \in S((y\ f^{-1}\ x_1 \wedge y\ f^{-1}\ x_2) \rightarrow x_1 = x_2)$ by definition of inverse relation.
8. This means the x in Line 4 is unique.
9. Thus f^{-1} is a function by definition of function.

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Proof: backward direction

(Need to prove: if f^{-1} is a function then f is injective.)

10. Assume f^{-1} is a function:
11. Suppose f is not injective.
12. Then $\exists y \in T, \exists x_1, x_2 \in S \ (x_1 \ f \ y \land x_2 \ f \ y \land x_1 \neq x_2)$ by definition of injective.
13. Thus $\exists y \in T, \exists x_1, \exists x_2 \in S \ (y \ f^{-1} \ x_1 \land y \ f^{-1} \ x_2 \land x_1 \neq x_2)$ by definition of the inverse relation.
14. Therefore f^{-1} is not a function. Contradiction.
15. Therefore f is injective.

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Proof cont'd

(Need to prove: if f^{-1} is a function then f is surjective.)

16. Assume f^{-1} is a function:
17. Suppose f is not surjective:
18. Then $\exists y \in T, \forall x \in S \sim(f(x) = y)$ by definition of surjective.
19. Thus $\exists y \in T, \forall x \in S \sim(y\ f^{-1}\ x)$ by definition of the inverse relation.
20. Therefore f^{-1} is not a function. Contradiction.
21. Therefore f is surjective.
22. Hence f is bijective, by Lines 15,21.
23. Hence f is bijective iff f^{-1} is a function. ■

7.3. Composition

Proposition 7.3.1

Let $f : S \to T$ be a function. Let $g : T \to U$ be a function. The composition of f and g, $g \circ f$, is a function from S to U.

Note that $(g \circ f)(x)$ means $g(f(x))$.

Proof.

1. Let $f : S \to T$ be a function
2. Let $g : T \to U$ be a function.
3. Therefore $g \circ f$ is a relation on the sets S and U by Definition 8.2.8 (Composition of relations).
4. Therefore $\forall x \in S, \exists! y \in T \ (x \ f \ y)$ by Definition 7.1.1.
5. Therefore $\forall y \in T, \exists! z \in U \ (y \ g \ z)$ by Definition 7.1.1.
6. Therefore $\forall x \in S, \exists! z \in U \ (x \ (g \circ f) \ z)$ by Steps (4), (5) and by Definition 8.2.8
7. Therefore $g \circ f$ is a function from S to U by Steps (3) and (6). ■

7.3.2. Identity

Definition 7.3.2 (Identity function)

Given a set A, define a function $\mathcal{I}_A$ from A to A by:

$$\forall x \in A \ (\mathcal{I}_A(x) = x)$$

This is the identity function on A.

Proposition 7.3.3

Let $f : A \to A$ be an injective function on A. Then $f^{-1} \circ f = \mathcal{I}_A$.

Proof omitted.

Notice that $f \circ f^{-1} = f^{-1} \circ f = \mathcal{I}_A$ if, and only if, f^{-1} is also a function. That is, if, and only if, f is bijective according to Proposition 7.2.4.

Generalization

So far, our functions accept only one argument. We can easily generalize functions to accept two arguments simply by making the domain a Cartesian product.

For example, let $f : A \times B \to C$ be a function. Then the argument to f is an ordered pair (a, b), where $a \in A$ and $b \in B$. That is, we write $f((a, b))$; which we simplify to $f(a, b)$.

Likewise, if $g : A \times B \times C \to D$ is a function, then we will write $g(a, b, c)$ instead of $g((a, b, c))$. In this way, we can allow functions to accept any finite number of arguments.

Finally, we may allow functions to return "multiple values" by making the co-domain a Cartesian product. Example:

Define $h : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by $\forall x \in \mathbb{Z},\ h(x) = (x^2, x + 3)$.

7.4.1. Exercises

Exercise 1

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$\forall x \in \mathbb{R}, \quad f(x) = 4x - 1$$

Is f one-to-one (injective)? Prove or give a counter-example.

To prove one-to-one, according to Definition 7.2.1, we need to show:

$$\forall x_1, x_2 \ \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2$$

Proof:

1. For any $x_1, x_2 \in \mathbb{R}$:
2. If $f(x_1) = f(x_2)$:
3. Then $4x_1 - 1 = 4x_2 - 1$, by definition of f.
4. Then $x_1 = x_2$, by basic algebra.
5. Hence f is indeed one-to-one. $\blacksquare$

Exercise 2

Define $g : \mathbb{Z} \to \mathbb{Z}$ by

$$\forall x \in \mathbb{Z}, \quad g(x) = x^2.$$

Show that g is not injective.

1. Clearly, $g(3) = 9 = g(-3)$.
2. But $3 \neq -3$.
3. Hence g is not injective.

Exercise 3

Define $f : \mathbb{R} \to \mathbb{R}$ by $\forall x \in \mathbb{R}, \quad f(x) = 4x - 1.$

Define $h : \mathbb{Z} \to \mathbb{Z}$ by $\forall n \in \mathbb{Z}, \quad h(n) = 4n - 1.$

Is f onto? Is h onto? Prove or give a counter-example.

From Definition 7.2.2, to prove that a function $F : X \to Y$ is onto, we need to show that:

$$\forall y \in Y, \ \exists x \in X \ (F(x) = y)$$

Proof: (by Construction)

1. Take any $y \in \mathbb{R}$.
2. Let $x = (y + 1)/4$.
3. Then $x \in \mathbb{R}$ because real numbers are closed under addition and division.
4. Thus $f(x) = f\left(\frac{y+1}{4}\right)$, by substitution.
5. $= 4\left(\frac{y+1}{4}\right) - 1$, by definition of f.
6. $= (y + 1) - 1 = y$, by basic algebra.
7. Thus f is onto. ■

However, for h, if we attempt the same proof, we will arrive at $n = \frac{m+1}{4}$.

This n may not be an integer, even if m is. For example, let $m = 0$, then $n = \frac{1}{4}$. This allows us to give a counter-example.

Disproof by counterexample:

1. Let $y = 0$.
2. (Claim: there is no $n \in \mathbb{Z}$ such that $h(n) = y$)
3. Suppose $\exists n \in \mathbb{Z}$ such that $h(n) = y$:
4. Then $4n - 1 = 0$, by substitution.
5. Thus $4n = 1$, by basic algebra.
6. Thus $4 \mid 1$, by definition of divisibility.
7. Contradiction.
8. Hence, $\exists y \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}\ (h(n) \neq y)$.
9. Hence h is not onto. ■

7.5. Summary

- A function is a special case of a relation.
- Important function properties of functions are: injective, surjective and bijective.
- The inverse of a function exists only iff it is bijective.
- Functions may be composed, just like relations.