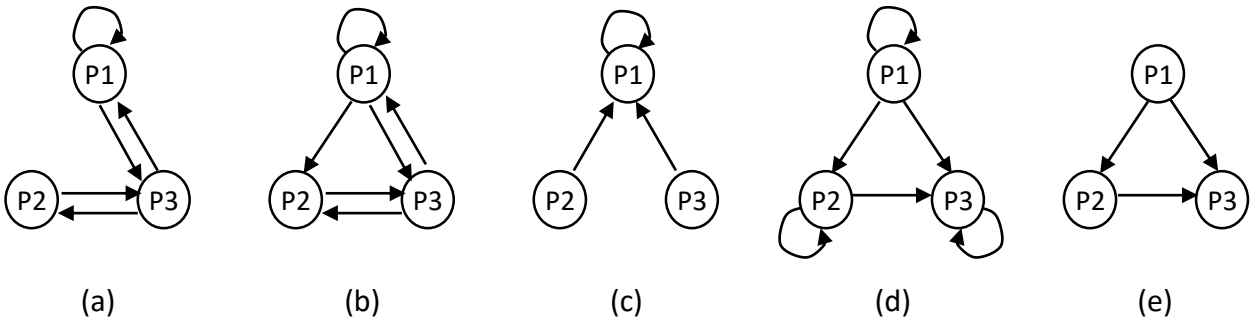
Tutorial 10
Graphs I

1 Exploration

Read the document "IdolRank" posted on the IVLE "Tutorials" workbin or the CS1231 website "Tutorials" page.

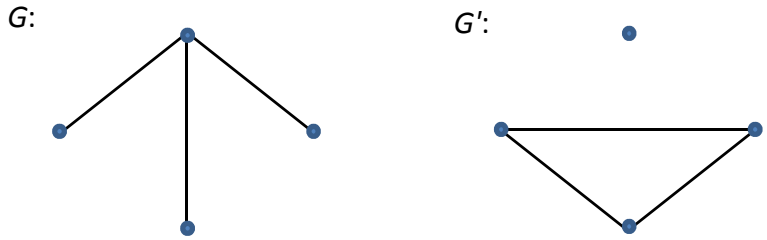
By hand or using a computer program, find out the winner of each of the five graphs below. $P1$, $P2$ and $P3$ represent three contestants, and an arrow from vertex x to vertex y indicates that x is the referee of y. The second graph is already solved in the above "IdolRank" document.



2 Tutorial questions

Definition 1. If G is a simple graph, the *complement* of G, denoted G', is obtained as follows: The vertex set of G' is identical to the vertex set of G. However, two distinct vertices v and w of G' are connected by an edge if, and only if, v and w are not connected by an edge in G.

The figure below shows a graph G and its complement G'.



A graph G and its complement G'.

Definition 2. A simple circuit (cycle) of length three is called a *triangle*.

Q1. (AY2016/17 Semester 1 Exam Question)
How many simple graphs on 3 vertices are there? In general, how many simple graphs are there on n ($n > 1$) vertices?

Q2. (AY2016/17 Semester 1 Exam Question)
Let G be a simple graph with n vertices where every vertex has degree at least $\left\lfloor \frac{n}{2} \right\rfloor$. Prove that G is connected.

Q3. Show that every simple graph with at least two vertices has two vertices of the same degree.

Q4. Prove that for any simple graph G with six vertices, G or its complement graph G' contains a triangle.

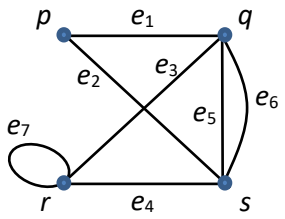
Q5. Show that in a connected simple graph any two longest paths have a vertex in common.

Q6. Answer the following questions on complete graph and Hamiltonian cycle.

a. How many Hamiltonian cycles are there in a complete graph $K_n (n \geq 3)$? In this question, the start vertex and direction of the cycle do not matter. Hence, there is only one Hamiltonian cycle in K_3.

b. Prove by induction on n that a complete graph K_n on $n \geq 3$ vertices contains a Hamiltonian cycle.

Q7. Given the following graph shown in the figure below.



a. Write the adjacency matrix $\mathbf{A}$ for the graph. Let the rows and columns be p, q, r and s.

b. Find $\mathbf{A^2}$ and $\mathbf{A^3}$.

c. How many walks of length 2 are there from p to q? From s to itself? List out all the walks.

d. How many walks of length 3 are there from r to s? From s to p? List out all the walks.

Q8. (AY2017/18 Semester 1 Exam Question)
Suppose you are given a pile of n stones. At each step, you are allowed to separate a pile of k stones into two piles of k_1 and k_2 stones. Obviously, $k_1 + k_2 = k$. On doing this, you earn $k_1 \times k_2$ dollars.

What is the maximum amount of money you can earn starting with a pile of n stones? Explain your answer.