Tutorial 1
Proofs and Logic

Tutorials are meant to reinforce the material taught in lecture. Therefore, please try these exercises before going to class. In doing so, you may discover gaps in your understanding. You will be asked to present your solutions in class, and this can help you see alternative solutions your classmates may have. Keep in mind that the goal of tutorials is *not* to answer every exercise, but to clarify doubts and reinforce concepts. Solutions to all tutorial exercises will be given in the following week. Tutorial attendance will be taken and contributes to 5% of the final grade.

1 Discussion questions

Discussion questions are meant for discussion on the IVLE Forum. You may try them on your own or discuss them with your classmates. No answers will be provided by us.

D1. The island of Wantuutrewan is inhabited by two types of people: knights who always tell the truth and knaves who always lie. Aaron comes to a fork in a road on the island. One branch leads to the ancient ruins he wants to visit, and the other branch leads deep into the deadly jungle. At the fork stand two natives, one is a knight and the other a knave, who know each other well. However, Aaron has no idea who is the knight and who is the knave.

Aaron wants to ask for direction to the ruins, but he is allowed to ask only one question to one of the two natives, and the natives, who understand English but cannot speak it, can only point with their fingers. What single question should Aaron ask so that he is sure to take the branch leading to the ruins and not to the jungle?

D2. Let p stand for the proposition "I went to Universal Studios Singapore" and q stand for "I rode the Battlestar Galactica". Express the following as natural English sentences. For part (f), express it in English before and after simplifying the given proposition.

(a) $\sim p$

(b) $p \vee q$

(c) $p \wedge \sim q$

(d) $p \rightarrow q$

(e) $\sim p \rightarrow \sim q$

(f) $\sim p \vee (p \wedge q)$

D3. A man has hidden his treasure somewhere on his property. He left a note in which he listed five statements (a-e below) and challenged the reader to use them to figure out the location of the treasure.

(a) If this house is next to a lake, then the treasure is not in the kitchen.

(b) If the tree in the front yard is an elm, then the treasure is in the kitchen.

(c) If the tree in the back yard is an oak, then the treasure is in the garage.

(d) The tree in the front yard is an elm or the treasure is burried under the flagpole.

(e) This house is next to a lake.

Where is the treasure hidden?

2 Tutorial questions

Q1. *Knights and Knaves*

You visit Wantuutrewan island (see discussion question 1 above) and have the following encounters with natives.

(a) Two natives A and B address you as follows:
A says: Both of us are knights.
B says: A is a knave.
What are A and B?

(b) Another two natives C and D approach you but only C speaks:
C says: Both of us are knaves.
What are C and D?

(c) You then encounter natives E and F.
E says: F is a knave.
F says: E is a knave.
How many knights and knaves are there?

Part (a) has been solved for you (see below). Following the same style of numbering and providing justification at every step, solve parts (b) and (c).

(a) Answer: A is a knave and B is a knight.

Proof (by Contradiction).

1. If A is a knight, then:
   1.1. What A says is true. *(by definition of knight)*
   1.2. $\therefore$ B is a knight also. *(that's what A said)*
   1.3. $\therefore$ What B says is true. *(by definition of knight)*
   1.4. $\therefore$ A is a knave. *(that's what B said)*
   1.5. $\therefore$ A is not a knight. *(a knave cannot also be a knight)*
   1.6. $\therefore$ Contradiction. *(contrary to the assumption that A is a knight)*
2. Therefore A is not a knight. *(by the contradiction rule)*
3. $\therefore$ A is a knave. *(since A is either a knight or a knave, but not both)*
4. $\therefore$ What B says is true. *(B said A was a knave, which we now know to be true)*
5. $\therefore$ B is a knight. *(by definition of knight)* ■

Remarks:

- Once we reach the contradiction in line 1.6., whatever statements we have derived under the current assumption (ie. that A is a knight, line 1.) are useless since we now know that the assumption is false. However, this does not mean that those statements are false either. Rather, their truth values are unknown.
- It is tempting to say "Contradiction" upon reaching line 1.4.. However, this is not valid because contradiction requires: $P \wedge \sim P$, but "knave" is not the negation of "knight". Thus the next line is needed.
- Note also that line 2. is not indented, because it does not depend on the assumption in line 1.

Q2. Simplify each of the following propositions using Theorem 2.1.1 (Epp) into propositions with only negation, conjunction and disjunction. Supply a justification for each step (eg: "by the commutative law").

(a) $(p \vee \sim q) \to q$

(b) $p \to (q \to r)$

(c) $(p \to q) \to r$

Prove, or disprove, that $(p \to (q \to r))$ is logically equivalent to $((p \to q) \to r)$.

Q3. Use Theorem 2.1.1 (Epp) to verify the logical equivalences for the following parts. Supply a reason for each step (eg: "by the commutative law").

a. $\sim(p \vee \sim q) \vee (\sim p \wedge \sim q) \equiv \sim p$.

b. $(p \wedge \sim(\sim p \vee q)) \vee (p \wedge q) \equiv p$.

Q4. Some of the arguments below are valid, whereas others exhibit the converse or the inverse error. Use symbols to write the logical form of each argument. If the argument is valid, identify the rule of inference that guarantees its validity. Otherwise, state whether the converse or the inverse error is made.

a. Sandra knows Java and Sandra knows C++.
$\therefore$ Sandra knows C++.

b. If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.
Neither of these two numbers is divisible by 6.
$\therefore$ The product of these two number is not divisible by 6.

c. If there are as many rational numbers as there are irrational numbers, then the set of all irrational numbers is infinite.
The set of all irrational numbers is infinite.
$\therefore$ There are as many rational numbers as there are irrational numbers.

d. If I get a Christmas bonus, I'll buy a stereo.
If I sell my motorcycle, I'll buy a stereo.
$\therefore$ If I get a Christmas bonus or I sell my motorcycle, then I'll buy a stereo.

An integer n is said to be *odd* if it can be written as $n = 2k + 1$; and *even* if it can be written as $n = 2k$, for some $k \in \mathbb{Z}$.

Examples of even integers: $-6, 0, 10, 2^k$ where k is any positive integer.
Examples of odd integers: $-7, 1, 17, 3^k$ where k is any positive integer.

Every integer is either odd or even, but not both (Theorem 4.6.2 (Epp)).

Q5. Prove Lemma 1:

Lemma 1: The product of any two odd integers is an odd integer.

Q6. I.M. Smart, your CS1231 classmate, came across this question:

Prove that if a, b, c are integers such that $a^2 + b^2 = c^2$, then a, b cannot both be odd.

(a) Smart attempts to prove the above:

Proof (by Contradiction).

1. Suppose a, b are both odd.
2. Then $\exists m, n \in \mathbb{Z}$ such that $a = 2m + 1$, and $b = 2n + 1$.
3. Then $a^2 + b^2 = (2m+1)^2 + (2n+1)^2 = 4m^2 + 4m + 4n^2 + 4n + 2 = c^2$.
4. Then $c = \sqrt{4m^2 + 4m + 4n^2 + 4n + 2}$.
5. But the right hand side is not an integer.
6. This contradicts the fact that c is an integer.
7. Hence a, b cannot both be odd. ■

Explain why Smart's proof is incomplete.

(b) Not to be left out, Aiken also tries to prove the statement:

Proof (by Contraposition).

1. (Want to prove: if a, b are both odd, then $a^2 + b^2 \neq c^2$)
2. Suppose a, b are both odd.
3. Then $\exists m, n \in \mathbb{Z}$ such that $a = 2m + 1$, and $b = 2n + 1$.
4. Then $a^2 + b^2 = (2m+1)^2 + (2n+1)^2 = 4m^2 + 4m + 4n^2 + 4n + 2$.
5. This number is even, and has remainder 2 when divided by 4.
6. Now, c is either odd or even.
7. <u>Case 1: c is odd</u>
   - 7.1. Then c^2 is odd.
   - 7.2. Then $c^2 \neq a^2 + b^2$, because the right hand side is even.
8. <u>Case 2: c is even</u>
   - 8.1. Then c^2 is even.
   - 8.2. Then $\exists k \in \mathbb{Z}$ such that $c = 2k$.
   - 8.3. Then $c^2 = 4k^2$.
   - 8.4. But this has remainder 0 when divided by 4.
   - 8.5. Hence $c^2 \neq a^2 + b^2$, since the right hand side has remainder 2 when divided by 4 (from Line 5.).
9. In all cases, $c^2 \neq a^2 + b^2$.
10. Therefore, by Contraposition, the original statement is true. ■

Is Aiken's proof correct? If so, how can it be improved? If not, where is it wrong?

Q7. Prove that $\exists! x \in \mathbb{R}, \exists! y \in \mathbb{R}$ such that $x^2 + y^2 = 0$.