Tutorial 3
Number Theory and Mathematical Induction

1 Discussion questions

Discussion questions are meant for discussion on the IVLE Forum. You may try them on your own or discuss them with your classmates. No answers will be provided by us.

D1. Slide 19 of `Week4_NumberTheory1.pdf` showed an application of prime numbers to encode two nonnegative integers m, n into a single integer s. The basic idea was to let $s = 2^m 3^n$. Explain how you can extend this idea to handle negative values of m and n.

D2. Slides 42,43 of `Week1_Proofs.pdf` gave a proof, by contradiction, that $\sqrt{2}$ is irrational. The contradiction is based on the fact that $\sqrt{2} = \frac{m}{n}$ is reduced to its lowest terms. Instead of this, can you use the Unique Prime Factorization Theorem to derive a different contradiction?

D3. Explain why $\log_5(2)$ is irrational.

2 Tutorial questions

> Define $\mathbb{Z}_{\geq k} = \{\ k, k+1, k+2, k+3, \ldots\ \}$ to be the set of integers starting from integer k.
>
> For example, $\mathbb{Z}_{\geq 5} = \{\ 5, 6, 7, 8, \ldots\ \}$.

Q1. Give an example to show that if d is not prime and n^2 is divisible by d, then n need not be divisible by d. Assume $n \in \mathbb{Z}$, and $d \in \mathbb{Z}^+$.

Q2. Prove that for all prime numbers a, b and c, $a^2 + b^2 \neq c^2$.
Hint: $a^2 = c^2 - b^2 = (c + b)(c - b)$.

Q3. I.M. Smart attempts to prove the following statement by Mathematical Induction. If the proof is wrong, correct it; otherwise improve it.

$$\forall n \in \mathbb{N},\ 3 \mid (n^3 + 44n).$$

Recall that $\mathbb{N}$ is defined to be the set of natural counting numbers, ie. $\mathbb{N} = \{ 0, 1, 2, 3, 4, 5, \ldots \}$.

Proof. (Proposed proof)

1. Note that $1^3 + 44(1) = 45$ which is divisible by 3.
2. So the statement is true for $n = 1$.
3. Now suppose the statement is true for some natural number k.
4. Then $k^3 + 44k$ is divisible by 3.
5. Therefore $(k+1)^3 + 44(k+1)$ is divisible by 3.
6. So by Mathematical Induction, the statement is true for all natural numbers. ■

Q4. Consider a group of n people, each of whom shakes hands exactly once with everybody else in the group. No one shakes his own hand. Let $S(n)$ be the total number of handshakes in any group of n people. Prove by Mathematical Induction that $\forall n \in \mathbb{Z}^+,\ S(n) = \frac{n(n-1)}{2}$.

Q5. Prove by Mathematical Induction that every positive integer $n \geq 12$ can be written as a linear combination of 4 and 5 using non-negative weights; that is,

$$\forall n \in \mathbb{Z}_{\geq 12},\ \exists x, y \in \mathbb{N} \text{ such that } n = 4x + 5y.$$

Note that the weights x, y are natural numbers; negative x, y are not allowed.

Q6. Let's make formal what we already know about the decimal representation of a number.

> **Definition:** Given any nonnegative integer n, the **decimal representation** of n is an expression of the form
>
> $$d_k d_{k-1} \cdots d_2 d_1 d_0,$$
>
> where k is a nonnegative integer; $d_0, d_1, d_2, \ldots, d_k$ (called the **decimal digits** of n) are integers from 0 to 9 inclusive; $d_k \neq 0$ unless $n = 0$ and $k = 0$; and
>
> $$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0.$$
>
> (For example, $2{,}503 = 2 \cdot 10^3 + 5 \cdot 10^2 + 0 \cdot 10 + 3$.)

Observe that

$$\begin{aligned}
7524 &= 7 \cdot 1000 + 5 \cdot 100 + 2 \cdot 10 + 4 \\
&= 7(999 + 1) + 5(99 + 1) + 2(9 + 1) + 4 \\
&= (7 \cdot 999 + 7) + (5 \cdot 99 + 5) + (2 \cdot 9 + 2) + 4 \\
&= (7 \cdot 999 + 5 \cdot 99 + 2 \cdot 9) + (7 + 5 + 2 + 4) \\
&= (7 \cdot 111 + 5 \cdot 11 + 2) \cdot 9 + (7 + 5 + 2 + 4) \\
&= (\text{an integer divisible by } 9) + (\text{the sum of the digits of } 7524).
\end{aligned}$$

Since the sum of the digits of 7524 ($= 18 = 2 \cdot 9$) is divisible by 9, the last line above shows that 7524 can be written as a sum of two terms, both of which are divisible by 9. From Theorem 4.1.1 (Linear Combination), this means that 7524 is divisible by 9.

Generalize the above to prove that any positive integer n is divisible by 9 if the sum of its digits is divisible by 9.

(Hint:) Use, without proving, the fact that for any positive integer k,

$$\begin{aligned}
10^k &= \underbrace{99\ldots9}_{k \text{ of these}} + 1 \\
&= 9 \cdot 10^{k-1} + 9 \cdot 10^{k-2} + \cdots + 9 \cdot 10^1 + 9 \cdot 10^0 + 1
\end{aligned}$$

Q7. (AY2015/16 Sem1, midterm) Find a positive integer n such that: (i) its prime factorization contains no repeated prime factors; and (ii) for any prime p, $p \mid n \longleftrightarrow (p-1) \mid n$.

Be sure to clearly explain and justify how you obtain n. The following list of primes under 100 may be useful.

2,	3,	5,	7,	11,	13,	17,	19,	23,	29,
31,	37,	41,	43,	47,	53,	59,	61,	67,	71,
73,	79,	83,	89,	97.					