Tutorial 7
Partial Orders and Functions

1 Discussion questions

Discussion questions are meant for discussion on the IVLE Forum. You may try them on your own or discuss them with your classmates. No answers will be provided by us.

D1. Let S be the set of all strings over the alphabet $\mathcal{A} = \{a, b\}$, ie. an element of S is a sequence of characters, each of which is either a or b. For example: $abb, bababba, bbaa$. Define a relation R on S by:

$$\forall s, t \in S,\ s\ R\ t \ \leftrightarrow\ l(s) \leq l(t)$$

where $l(x)$ denotes the length (the number of characters) of x. Is R anti-symmetric? Prove or give a counter-example.

D2. Which of the following is a function? If it is not a function, explain why not.

(a) Define $f : \mathbb{Z} \to \mathbb{Z}$ by $\forall z \in \mathbb{Z},\ f(z) = \begin{cases} 1, & \text{if } 2 \mid z, \\ 2, & \text{if } 3 \mid z. \end{cases}$

(b) Define $f : \mathbb{Z} \to \mathbb{Z}$ by $\forall z \in \mathbb{Z},\ f(z) = \begin{cases} 1, & \text{if } 2 \mid z, \\ 2, & \text{if } 2 \nmid z. \end{cases}$

(c) Define $f : \mathbb{R} \to \mathbb{Z}$ by $\forall x \in \mathbb{Z},\ f(x) = 2x$.

(d) Define $f : \mathbb{Z} \to \mathbb{R}$ by $\forall x \in \mathbb{Z},\ f(x) = 2x$.

D3. Define a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ by $\forall x, y \in \mathbb{Z},\ f(x, y) = \frac{x+y}{3}$. Find three distinct pre-images of 2.

2 Tutorial questions

Q1. Let $S = \{0, 1\}$ and define the relation $\mathcal{R}$ on $S \times S$ as follows:

$$\forall (a, b), (c, d) \in S \times S\ \big(\ (a, b)\ \mathcal{R}\ (c, d) \ \leftrightarrow\ (a \leq c) \wedge (b \leq d)\ \big)$$

where $\leq$ denotes the usual "less than or equal to" relation for real numbers.

(a) Prove that $\mathcal{R}$ is a partial order.

(b) Draw the Hasse diagram for $\mathcal{R}$.

(c) Find the maximum, maximal, minimum and minimal elements.

(d) Is $(S \times S,\ \mathcal{R})$ well-ordered?

Q2. Let $\mathcal{R}$ be a binary relation defined on a set A. From the lecture notes, we have already defined *anti-symmetry*:

<u>Definition 1:</u> $\mathcal{R}$ is said to be *anti-symmetric* iff

$$\forall x, y \in A\ ((x\ \mathcal{R}\ y\ \wedge\ y\ \mathcal{R}\ x)\ \rightarrow\ x = y)$$

We make a new definition:

<u>Definition 2:</u> $\mathcal{R}$ is said to be *asymmetric* iff

$$\forall x, y \in A \ (x \ \mathcal{R} \ y \ \rightarrow \ y \not\mathcal{R} \ x)$$

Using the set $A = \{1, 2, 3\}$,

(a) Find a relation on A that is both asymmetric and anti-symmetric.

(b) Find a relation on A that is *not* asymmetric but anti-symmetric.

(c) Find a relation on A that is asymmetric but *not* anti-symmetric.

(d) Find a relation on A that is neither asymmetric nor anti-symmetric.

Q3. Suppose a binary relation $\mathcal{R}$ on a set A is reflexive, symmetric, transitive and anti-symmetric. What can you conclude about $\mathcal{R}$? Explain.

Q4. Lexicographic Order: To compare two English words ("strings"), you compare their letters one by one from left to right. If they have been the same to a certain point, and one word runs out of letters, then the shorter word comes first. For example, "run" comes before "runner". If all letters up to a certain point are the same, but the next letter differs, then the word with the "smaller" differing letter (in the usual alphabetic sense) comes first. For example, "runner" comes before "runway"; also, "chicken" comes before "dog". This is formally defined in Theorem 8.5.1 (Epp).

Theorem 8.5.1

Let A be a set with a partial order relation R, and let S be a set of strings over A. Define a relation $\preceq$ on S as follows:

For any two strings in S, $a_1 a_2 \cdots a_m$ and $b_1 b_2 \cdots b_n$, where m and n are positive integers,

1. If $m \leq n$ and $a_i = b_i$ for all $i = 1, 2, \ldots, m$, then
$$a_1 a_2 \cdots a_m \preceq b_1 b_2 \cdots b_n.$$
2. If for some integer k with $k \leq m$, $k \leq n$, and $k \geq 1$, $a_i = b_i$ for all $i = 1, 2, \ldots, k-1$, and $a_k \neq b_k$, but $a_k \ R \ b_k$ then
$$a_1 a_2 \cdots a_m \preceq b_1 b_2 \cdots b_n.$$
3. If ε is the null string and s is any string in S, then $\epsilon \preceq s$.

If no strings are related other than by these three conditions, then $\preceq$ is a partial order relation.

Let $A = \{a, b\}$ be our alphabet (instead of the usual 26-letter alphabet of the English language). And let A have the usual partial order defined on it, ie. the letter a comes before b. Now define S to be the set of all strings over alphabet A; that is, S contains strings made up of letters a and b. Define the partial order $\preceq$ on S to be the lexicographic order specified in Theorem 8.5.1 (Epp). Determine which of the following statements are true; and for each true statement, state which of the three rules in Theorem 8.5.1 (Epp) makes the statement true.

(a) $aab \preceq aaba$

(b) $bbab \preceq bba$

(c) $\varepsilon \preceq aba$

(d) $aba \preceq abb$

(e) $bbab \preceq bbaa$

(f) $ababa \preceq ababaa$

(g) $bbaba \preceq bbabb$

Q5. Let $A = \{a, b\}$ and S be the set of all strings over alphabet A. Define $C : S \to S$ by

$$C(s) = as, \quad \text{for all } s \in S.$$

C is called *concatenation* by a on the left.

(a) Is C one-to-one? Prove or give a counterexample.

(b) Is C onto? Prove or give a counterexample.

Note: For any two strings s and t, $s = t$ if, and only if, they have the same length (same number of letters), and all corresponding letters are equal. For example, $abaab = abaab$, $aba \neq abaab$, $bab \neq baa$.

Q6. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by:

$$f(x) = x + 3, \quad \text{and} \quad g(x) = -x, \quad \text{for all } x \in \mathbb{R}.$$

Find:

(a) $g \circ f$

(b) $(g \circ f)^{-1}$

(c) g^{-1}

(d) f^{-1}

(e) $f^{-1} \circ g^{-1}$.

Which two are the same?

Q7. Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Prove each of the following statements, or disprove it with a counter-example.

(a) If $g \circ f$ is injective, then g is injective.

(b) If $g \circ f$ is injective, then f is injective.

(c) If $g \circ f$ is surjective, then g is surjective.

(d) If $g \circ f$ is surjective, then f is surjective.

Q8.

(a) Find a function $f : \mathbb{Z} \to \mathbb{Z}$ that is injective but not surjective.

(b) Find a function $g : \mathbb{Z} \to \mathbb{Z}$ that is surjective but not injective.