Tutorial 8
Counting and Probability I

1 Discussion questions

Discussion questions are meant for discussion on the IVLE Forum. You may try them on your own or discuss them with your classmates. No answers will be provided by us.

D1. A box contains three blue balls and seven white balls. One ball is drawn, its colour recorded, and it is returned to the box. Then another ball is drawn and its colour is recorded as well.

a. What is the probability that the first ball drawn is blue and the second is white?
b. What is the probability that both balls drawn are white?
c. What is the probability that the second ball drawn is blue?

D2. Answer the following:

a. What is the probability that a randomly chosen positive three-digit integer is a multiple of 6?
b. What is the probability that a randomly chosen positive four-digit integer is a multiple of 7?

D3. Assuming that all years have 365 days and all birthdays occur with equal probability, what is the smallest value for n so that in any randomly chosen group of n people, the probability that two or more persons having the same birthday is at least $1/2$?

Write out the equation to solve for n and write a program to compute n.

(This is the well-known *birthday problem*, whose solution is counter-intuitive but true.)

2 Tutorial questions

Q1. Repeat discussion question D1, except that now after the first ball is drawn, it is not returned to the box.

Q2. In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed? Use a possibility tree to solve this problem.

Q3. Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls are consecutively numbered. What is the total number of ways this can be done?

(In the next tutorial, we will approach this problem using combination.)

Q4. There is an office building at HarbourFront with 50 floors. Each floor has 20 offices on the south side facing Sentosa, and 20 offices on the north side facing Mount Faber.

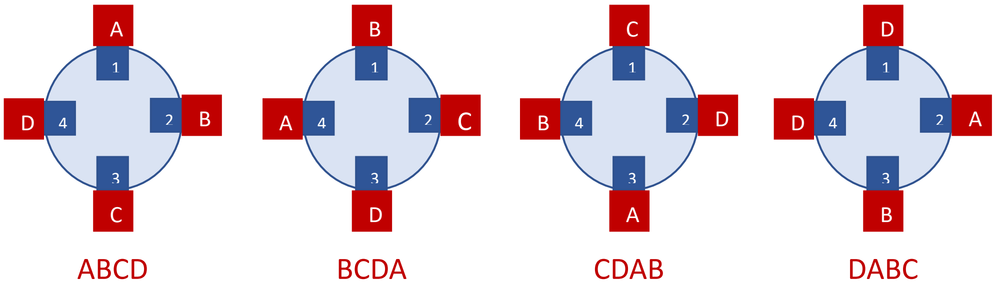
There are three night security guards who each check half the offices and make sure that all is secure and that the lights are turned out.

Avery checks all the offices on the north side on even floors and those on the south side on odd floors. Bruce checks all the offices on odd floors. Calson checks all the offices on the top 25 floors.

In the worst case, how many offices have their lights left on?

Q5. We have learned that the number of permutations of n distinct objects is $n!$, but that is on a straight line. If we seat four guests Anna, Barbie, Chistian and Dorcas on chairs on a straight line they can be seated in 4! or 24 ways.

What if we seat them around a circular table? Examine the figure below.



The four seating arrangements (clockwise from the top) $ABCD, BCDA, CDAB$ and $DABC$ are just a single permutation, as in each arrangement the persons on the left and on the right of each guest are still the same persons. Hence, these four arrangements are considered as one permutation.

This is known as *circular permutation*. The number of linear permutations of 4 persons is four times its number of circular permutation. Hence, for 4 persons, there are $4!/4$ or $3!$ ways of circular permutations. In general, the number of circular permutations of n objects is $(n-1)!$

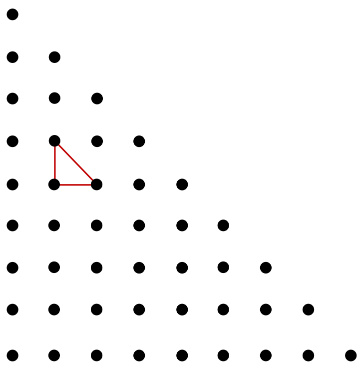
Answer the following questions:

a. In how many ways can 8 boys and 4 girls sit around a circular table, so that no two girls sit together?

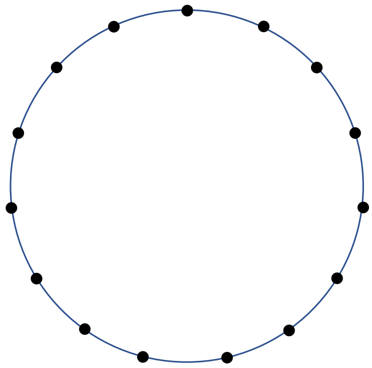
b. In how many ways can 6 people sit around a circular table, but Eric would not sit next to Danny?

c. In how many ways can 10 different beads be arranged to form a necklace? (Note that for a necklace, the clockwise and anti-clockwise arrangements are considered the same.)

Q6. The diagram below shows a grid of dots, where each dot has equal vertical and horizontal spacing from the others. A small 45-45-90 triangle is drawn. How many triangles congruent to this triangle, of any orientation, can be constructed from the dots in this grid?



Q7. On the circumference of a circle are 15 dots, evenly spaced out. How many combinations of 3 dots can we pick from these 15 dots that do not form an equilateral triangle?



Q8. *(AY2015/16 Semester 1 Exam Question)*
Prove that among any five points selected inside a square with a side length of 2 units, there always exists a pair of these points that are within $\sqrt{2}$ units of each other.

Q9. *(AY2016/17 Semester 1 Exam Question)*
Prove that if you randomly put 51 points inside a unit square, there are always three points that can be covered by a circle of radius $1/7$.