Tutorial 9
Counting and Probability II

1 Discussion questions

Discussion questions are meant for discussion on the IVLE Forum. You may try them on your own or discuss them with your classmates. No answers will be provided by us.

D1. How many one-to-one functions are there from a set with m elements to a set with n elements, where $m \leq n$?

D2. (Epp Exercise Set 9.7 Q16-17)
Think of a set with $m + n$ elements as composed of two parts, one with m elements and the other with n elements. Give a combinatorial argument to show that

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}, \ . \ . \ . \ \text{(A)}$$

where m and n are positive integers and r is an integer that is less than or equal to both m and n.

Using (A), prove that for all integers $n \geq 0$,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

D3. Suppose a random sample of 2 lighbulbs is selected from a group of 8 bulbs in which 3 are defective, what is the expected value of the number of defective bulbs in the sample? Let X represent the number of defective bulbs that occur on a given trial, where $X = 0, 1, 2$. Find $E[X]$.

2 Tutorial questions

Q1. *(AY2015/16 Semester 1 exam question)*
You wish to select five persons from seven men and six women to form a committee that includes at least three men.

a. In how many ways can you form the committee?

b. If you randomly choose five persons to form the committee, what is the probability that you will get a committee with at least three men?

Q2. Let's revisit Question 3 of Tutorial 8:

Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls are consecutively numbered. What is the total number of ways this can be done?

The answer given was:

1. For $k(1 \leq k \leq n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways.
2. Total number of ways $= \sum_{k=1}^{n}(n - k + 1) = \sum_{k=1}^{n} k = n(n + 1)/2$.

Now, let's use another approach to solve this. Draw crosses on the side of the boxes as shown below. What do you do with the crosses?

Q3. Find how many solutions there are to the following equation:

$$x_1 + x_2 + x_3 + x_4 = 30, \text{ where } x_i \text{ are integers and } x_i \geq 2.$$

(You may refer to Example 9.6.6 on page 589 of Epp's book for an example.)

Q4. *(AY2015/16 Semester 1 exam question)*

Let $A = 1, 2, 3, 4$. Since each element of $\mathcal{P}(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. Assuming each relation in $\mathcal{P}(A \times A)$ is likely to be chosen, what is the probability that a randomly chosen relation is

a. reflexive?

b. symmetric?

Q5. Show that

$$\binom{n}{k} = \sum_{p=k-1}^{n-1} \binom{p}{k-1}$$

Q6. Do the following questions using Binomial Theorem.

a. Using the result of an example shown in class, evaluate the following:

$$\binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \binom{9}{4} + \binom{9}{5} + \binom{9}{6} + \binom{9}{7} + \binom{9}{8}$$

b. Without using a calculator, find to the nearest dollar the amount that \$1000 will accrue to in 10 years at 4% p.a. compound interest.

c. Find the value of r if the co-efficients of x^r and x^{r+1} are equal in the binomial expansion of $(3x + 2)^{19}$.

Q7. a. A lottery game offers \$1 million to the grand prize winner, \$1000 to each of 100 second prize winners, \$500 to each of the 300 third prize winnners and \$10 to each of the 1000 consolation prize winners. The cost of the lottery is \$3 per ticket and 500,000 tickets were sold. What is the expected gain or loss of a ticket?

b. An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls?

Q8. Given an unbiased coin, what is the expected number of coin tosses to get five consecutive heads?

Q9. One urn contains 10 red balls and 25 green balls, and a second urn contains 22 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a loaded coin with probability 0.4 of landing heads up and probability 0.6 of landing tails up. If the coin lands heads up, the first urn is chosen; otherwise, the second urn is chosen. Then a ball is picked at random from the chosen urn.

a. What is the probability that the chosen ball is green?

b. If the chosen ball is green, what is the probability that it was picked from the first urn?