CS1231S Assignment #1

AY2024/25 Semester 1

Deadline: Monday, 16 September 2024, 1:00pm

ANSWERS

IMPORTANT: Please read the instructions below carefully.

This is a graded assignment worth 10% of your final grade. There are **six questions** (an admin Q0 and six task questions Q1–6) with a total score of 40 marks. Please work on it by yourself, not in a group or discussion/in collaboration with anybody. Anyone found committing plagiarism (submitting other's work as your own), or sending your answers to others, or other forms of academic dishonesty, will be penalised with a straight zero for the assignment, and reported to the school. Please see SoC website "Preventing Plagiarism" https://www.comp.nus.edu.sg/cug/plagiarism/.

You must submit your assignment to **Canvas > Assignments > Assignment 1 submission** before the deadline. **Late submission will NOT be accepted**. We have set the closing time of submission to slightly after 1pm to give you a few minutes of grace, but in your mind, you should treat **1pm** as the deadline. If you think you might be too busy on the day of the deadline, please submit earlier. Also, avoid submitting in the last few minutes, as the system may get sluggish due to overload and you will miss the deadline, and we will not extend the deadline for you.

The question paper is on **Canvas > Files > Assignment 1**. In the same folder you can find the template submission files **assign1_template.docx** and **assign1_template.pdf**. Download one of these template files. You may proceed in one of these ways: type your answers in assign1_template.docx and convert it into a pdf file, or use a pdf editor and type your answers directly into assign1_template.pdf. You must rename the pdf file to **Axxxxxxxx.pdf** before you submit it, where Axxxxxxxx is your student number.

Note that we are using a software to scan your shaded Student Number, so the pdf file you submit must have the Student Number box (refer to the template file) at the exact position. Also, the answers you write must fall within the boxes at those fixed locations in the template file. So, **do not use your phone camera** to scan your file as it will likely not have the same pagination and positioning as the template files we provide.

You are to submit **a SINGLE pdf file**. You may submit multiple times, but only the last submitted file will be graded.

Note the following:

- Shade your **Student Number** correctly, as instructed on the template file.
- As this is an assignment given well ahead of time, we expect you to work on it early. You should submit **polished work**, not answers that are untidy or appear to have been done in a hurry, for example, with scribbling and cancellation all over the places. Marks may be deducted for untidy work.
- It is always good to be clear and leave no gap so that the marker does not need to make guesswork on your answer. When in doubt, the marker would usually not award the mark.
- Write/type your answers within the boxes given in the template file.
- Do not use any methods that have not been covered in class. When using a theorem or result that has appeared in class (lectures or tutorials), please quote the theorem number/name, the lecture and slide number, or the tutorial number and question number in that tutorial, failing which marks may be deducted. Remember to use numbering and give justification for important steps in your proof, or marks may be deducted.
- **[Very important!]** In the past, some students submitted a single-page file or even a blank file, or did not submit anything, without realising it! It is your responsibility to check that your submitted file is complete. We will not allow any submission after the deadline.

If you need any clarification about this assignment, please do NOT email us or post on telegram, but post on **QnA "Assignments"** topic so that everybody can read the answers to the queries.

Question 0. (Total: 2 marks)
Check that …

- you have submitted a pdf file with your Student Number as the filename. [1 mark]
- you have written both your name and tutorial group number (eg: T02) at the top of the first page of your file. (If you miss either one you will not be given any mark.) [1 mark]

Question 1. Propositional logic (Total: 7 marks)

(a) Prove the following statements using truth table for part (i) and Theorem 2.1.1 for part (ii). For truth tables, you may use **T** for true and **F** for false. For Theorem 2.1,1, you must write **true** and **false** for tautology and contradiction respectively.

(i) $p \wedge (\sim p \vee q) \equiv p \wedge q$ [1 mark]

(ii) $p \vee (\sim p \wedge q) \equiv p \vee q$ [2 marks]

The above are variants of the absorption laws in Theorem 2.1.1. To distinguish them from the absorption laws, we shall call them the "variant absorption laws" and you may cite them in your work from now on.

(b) Simplify the statement form below in no more than 9 steps. Make sure you do not skip any step, and every step must be justified by a law. Do not combine two steps of the same law in a single step. Use **true** and **false** for tautology and contradiction respectively. (This question checks that you apply the laws rigorously and cite them correctly, so we will be strict in our grading.) [4 marks]

$$(\boldsymbol{p} \wedge (\boldsymbol{p} \rightarrow \boldsymbol{r} \vee \boldsymbol{q})) \wedge (\boldsymbol{r} \rightarrow \boldsymbol{q})$$

Answers:

(a) (i)

p	q	$\sim p \vee q$	$p \wedge (\sim p \vee q)$	$p \wedge q$
T	T	T	T	T
T	F	F	F	F
F	T	T	F	F
F	F	T	F	F

(ii) $p \vee (\sim p \wedge q) \equiv (p \vee \sim p) \wedge (p \vee q)$ (by distributive law)
$\equiv \textbf{true} \wedge (p \vee q)$ (by negation law)
$\equiv (p \vee q) \wedge \textbf{true}$ (by commutative law) ← check if this step is missing?
$\equiv p \vee q$ (by identity law)

(b) $(p \wedge (p \rightarrow r \vee q)) \wedge (r \rightarrow q)$
$\equiv \big(p \wedge (\sim p \vee (r \vee q))\big) \wedge (r \rightarrow q)$ (by implication law) step 1
$\equiv \big(p \wedge (r \vee q)\big) \wedge (r \rightarrow q)$ (by variant absorption law) step 2
$\equiv \big(p \wedge (r \vee q)\big) \wedge (\sim r \vee q)$ (by implication law) step 3
$\equiv p \wedge \big((r \vee q) \wedge (\sim r \vee q)\big)$ (by associative law) step 4 ← check if this step is missing?
$\equiv p \wedge \big((q \vee r) \wedge (\sim r \vee q)\big)$ (by commutative law) step 5
$\equiv p \wedge \big((q \vee r) \wedge (q \vee \sim r)\big)$ (by commutative law) step 6
$\equiv p \wedge \big(q \vee (r \wedge \sim r)\big)$ (by distributive law) step 7
$\equiv p \wedge (q \vee \textbf{false})$ (by negation law) step 8
$\equiv p \wedge q$ (by identity law) step 9

Question 2. Argument (Total: 6 marks)
Determine whether the following argument is valid or invalid, assuming that p, q, r and s are statement variables:

$$(p \wedge q) \rightarrow r$$
$$(r \vee s) \rightarrow p$$
$$r \wedge \sim p$$
$$\therefore s \vee q$$

You should begin your answer by stating "**Valid**" or "**Invalid**", followed by your proof. You are not allowed to show your truth table (partial or full) in your answer, though you may use truth table in your own rough work to help you derive your proof.

Answers

Argument is **valid**. There are no critical rows in the truth table, as it is impossible for all 3 premises to be true. Recall that an argument with premises $P1, P2, \cdots, Pn$ and conclusion K is valid if and only if the statement $(P1 \wedge P2 \wedge \cdots \wedge Pn) \rightarrow K$ is a tautology (tutorial 1 additional notes).

To prove that the argument is valid, that is, $\big((P1 \wedge P2 \wedge P3) \rightarrow K\big) \equiv true$, it suffices to prove that $(P1 \wedge P2 \wedge P3) \equiv false$.

1. Suppose not, that is, $(P1 \wedge P2 \wedge P3) \equiv true$.
2. Then $r \wedge \sim p$ (which is $P3$) is true.
   2.1. r is true (by specialization)
   2.2. $\sim p$ is true (by specialization) which means that p is false (by negation of true)
3. Then $r \vee s \equiv true \vee s \equiv s \vee true$ (by commutative law) $\equiv true$ (by universal bound rule).
4. Then $P2 \equiv (r \vee s) \rightarrow p \equiv true \rightarrow false \equiv false$ (by definition of conditional statement) which contradicts line 1.
5. Hence, $(P1 \wedge P2 \wedge P3) \equiv false$.
6. Therefore, $\big((P1 \wedge P2 \wedge P3) \rightarrow K\big) \equiv true$.

Question 3. Sets (Total: 6 marks)
No working/justification is required for this question.

(a) For each $i \in \{1,2,3,4\}$, define $A_i = \{ik + 1 : k \in \{0,1,2,3,4\}\}$. Write down B_2, B_3, B_4 in **roster notation** with the following properties, where B_1, B_2, B_3, B_4 are mutually disjoint sets:

$B_1 = A_1$, $\quad B_1 \cup B_2 = A_1 \cup A_2$,

$B_1 \cup B_2 \cup B_3 = A_1 \cup A_2 \cup A_3$, $\quad B_1 \cup B_2 \cup B_3 \cup B_4 = A_1 \cup A_2 \cup A_3 \cup A_4$.

Do not use ellipses ("$\cdots$") in your answer. [3 marks]

(b) For each $k \in \mathbb{Z}_{\geqslant 0}$, let $D_k = \{n \in \mathbb{Z}_{\geqslant 0} : k = mn \text{ for some } m \in \mathbb{Z}_{\geqslant 3}\}$. Write down each of the following sets in **roster notation**. [3 marks]
(i) D_2 (ii) D_3 (iii) D_6 (iv) D_{18} (v) $\bigcap_{k=6}^{14} D_k$ (vi) $\bigcup_{k=6}^{14} D_k$

Answers

(a) $B_2 = \{7,9\}$, $B_3 = \{10,13\}$, $B_4 = \{17\}$.
(For reference: $A_1 = \{1,2,3,4,5\}, A_2 = \{1,3,5,7,9\}, A_3 = \{1,4,7,10,13\}, A_4 = \{1,5,9,13,17\}$).

(b) (i) $D_2 = \emptyset$ (ii) $D_3 = \{1\}$ (iii) $D_6 = \{1,2\}$ (iv) $D_{18} = \{1,2,3,6\}$
(v) $\bigcap_{k=6}^{14} D_k = \{1\}$ (vi) $\bigcup_{k=6}^{14} D_k = \{1,2,3,4\}$

Question 4. Quantified statements on Sets (Total: 6 marks)
Let $A = \{-2, -1, 0, 1, 2\}, B = \{0, 1, 4\}$ and $C = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

For each of the parts below, determine whether the given statement is true or false. Start your answer by writing "True" or "False", followed by your proof or counterexample. The symbol ⇔ is the biconditional connective representing "if and only if".

(a) $\forall x \in C\ \big((x \in A) \Leftrightarrow (x^2 \in B)\big)$. [3 marks]

(b) $\exists x \in A\ \forall y \in A\ \big((x \neq 0) \wedge (xy \in B)\big)$. [3 marks]

Answers

(a) The statement is **true**. Proof:

1. Case 1: $x = -2, -1, 0, 1, 2$
    1.1. $x \in A$ is true.
    1.2. $(-2)^2 = 2^2 = 4; (-1)^2 = 1^2 = 1; 0^2 = 0$. So $x^2 \in B$ is true.
    1.3. Hence $(x \in A) \Leftrightarrow (x^2 \in B)$ is true.
2. Case 2: $x = -4, -3, 3, 4$
    2.1. $x \notin A$, hence $x \in A$ is false.
    2.2. $(-4)^2 = 4^2 = 16;\ (-3)^2 = 3^2 = 9$. So $x^2 \notin B$, hence $x^2 \in B$ is false.
    2.3. Hence $(x \in A) \Leftrightarrow (x^2 \in B)$ is true.
3. In all cases, $(x \in A) \Leftrightarrow (x^2 \in B)$ is true.

(b) The statement is **false**. Proof:

1. Case $x = -2$: Counterexample (pick one) $y = -1$, or $y = 1$, or $y = 2$. Then $xy \notin B$.
2. Case $x = -1$: Counterexample (pick one) $y = -2$, or $y = 1$, or $y = 2$. Then $xy \notin B$.
3. Case $x = 0$: $x \neq 0$ is false.
4. Case $x = 1$: Counterexample (pick one) $y = -2$, or $y = -1$, or $y = 2$. Then $xy \notin B$.
5. Case $x = 2$: Counterexample (pick one) $y = -2$, or $y = -1$, or $y = 1$. Then $xy \notin B$.
6. In all cases, $\exists x \in A\ \forall y \in A\ \big((x \neq 0) \wedge (xy \in B)\big)$ is false.

Question 5. Proof (Total: 7 marks)
Consider this statement:

For any integers x and y, if $x + y$ is even, then both x and y are even or both are odd.

The following is Aiken's proof:

"We prove by contradiction. Suppose $x + y$ is even and for x and y, one of them is even and the other is odd. Without loss of generality, we may assume that x is even and y is odd. Then, $x = 2k$ for some integer k (by definition of even number) and $y = 2m + 1$ for some integer m (by definition of odd number). So we have $x + y = 2k + 2m + 1 = 2(k + m) + 1$. Since $k + m$ is an integer (by closure of integers under addition), $x + y$ is therefore odd. This contradicts the assumption that $x + y$ is even."

In your answer, you may use the predicates $Even(n)$ to mean "n is even" and $Odd(n)$ to mean "n is odd". Also recall in Lecture 1 slide 27 Assumption 1: Every integer is even or odd, but not both. This means $\sim Even(n) \equiv Odd(n)$.

(a) Rewrite the given statement using mathematical notation. You answer should not contain any English words like "such that". [2 marks]

(b) Explain why the proof begins by assuming that one integer is even and the other is odd. You may skip the commutative laws in your explanation, and you may combine the same rule in consecutive steps into one step (eg: nagation law twice). (Your explanation should be in a form of a proof, rather than an English essay.) [4 marks]

(c) Explain what is meant by "without loss of generality" in this proof. [1 mark]

Answers

(a) $\forall x, y \in \mathbb{Z}\ \Big(Even(x+y) \rightarrow \big(Even(x) \wedge Even(y)\big) \vee \big(Odd(x) \wedge Odd(y)\big)\Big)$

(b) The given statement is in the form $\forall x, y \in \mathbb{Z}\ \big(P(x) \rightarrow Q(x)\big)$. To prove by contradiction, we begin by negating the statement into $\exists x, y \in \mathbb{Z}\ \big(P(x) \wedge {\sim}Q(x)\big)$.

$$
\begin{aligned}
{\sim}Q(x) &\equiv {\sim}\Big(\big(Even(x) \wedge Even(y)\big) \vee \big(Odd(x) \wedge Odd(y)\big)\Big) \\
&\equiv {\sim}\big(Even(x) \wedge Even(y)\big) \wedge {\sim}\big(Odd(x) \wedge Odd(y)\big) && \text{by De Morgan's law} \\
&\equiv \big({\sim}Even(x) \vee {\sim}Even(y)\big) \wedge \big({\sim}Odd(x) \vee {\sim}Odd(y)\big) && \text{by De Morgan's law twice} \\
&\equiv \big(Odd(x) \vee Odd(y)\big) \wedge \big({\sim}Odd(x) \vee {\sim}Odd(y)\big) && \text{by } {\sim}Even(n) \equiv Odd(n) \\
&\equiv \Big(\big(Odd(x) \vee Odd(y)\big) \wedge {\sim}Odd(x)\Big) \vee \Big(\big(Odd(x) \vee Odd(y)\big) \wedge {\sim}Odd(y)\Big) \\
& && \text{by distributive law twice} \\
&\equiv \Big({\sim}Odd(x) \wedge \big(Odd(x) \vee Odd(y)\big)\Big) \vee \Big({\sim}Odd(y) \wedge \big(Odd(x) \vee Odd(y)\big)\Big) \\
& && \text{by commutative law twice} \\
&\equiv \Big(\big({\sim}Odd(x) \wedge Odd(x)\big) \vee \big({\sim}Odd(x) \wedge Odd(y)\big)\Big) \\
&\qquad \vee \Big(\big({\sim}Odd(y) \wedge Odd(x)\big) \vee \big({\sim}Odd(y) \wedge Odd(y)\big)\Big) \\
& && \text{by distributive law twice} \\
&\equiv \Big(\big(Odd(x) \wedge {\sim}Odd(x)\big) \vee \big({\sim}Odd(x) \wedge Odd(y)\big)\Big) \\
&\qquad \vee \Big(\big({\sim}Odd(y) \wedge Odd(x)\big) \vee \big(Odd(y) \wedge {\sim}Odd(y)\big)\Big) \\
& && \text{by commutative law twice} \\
&\equiv \Big(false \vee \big({\sim}Odd(x) \wedge Odd(y)\big)\Big) \vee \Big(\big({\sim}Odd(y) \wedge Odd(x)\big) \vee false\Big) \\
& && \text{by negation law twice} \\
&\equiv \Big(\big({\sim}Odd(x) \wedge Odd(y)\big) \vee false\Big) \vee \Big(\big({\sim}Odd(y) \wedge Odd(x)\big) \vee false\Big) \\
& && \text{by commutative law} \\
&\equiv \big({\sim}Odd(x) \wedge Odd(y)\big) \vee \big({\sim}Odd(y) \wedge Odd(x)\big) && \text{by identity law twice} \\
&\equiv \big(Even(x) \wedge Odd(y)\big) \vee \big(Even(y) \wedge Odd(x)\big) && \text{by } {\sim}Even(n) \equiv Odd(n)
\end{aligned}
$$

The proof begins by considering $\big(Even(x) \wedge Odd(y)\big)$.

(c) There are 2 cases to consider: (i) x is even and y is odd; or (ii) x is odd and y is even. The proof for one case is the same as the proof for the other case, except for switching the roles of x and y.

Question 6. Proof on Sets (6 marks)

Prove or disprove the statements below. Do not use Venn diagrams. Start your answer by writing "True" or "False", followed by your proof or counterexample.

(a) For all sets A, B and C, $(A \cap B) \times (B \cap C) = (B \cap C) \times (A \cap B)$. [1 mark]

(b) For every set A, if $A \subseteq \emptyset$, then $A = \emptyset$. [2 marks]

(c) For all sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. (Note: $\mathcal{P}(X)$ is the power set of X.) [3 marks]

Answers:

(a) The statement is **false**. Cartesian product is not commutative, so there are many possible counterexamples. One counterexample: $A = \{1,2\}, B = \{2,3\}, C = \{1,3\}$.

Then $(A \cap B) \times (B \cap C) = \{2\} \times \{3\} = \{(2,3)\}$ but $(B \cap C) \times (A \cap B) = \{3\} \times \{2\} = \{(3,2)\}$.

(b) The statement is **true**. Proof:

1. Suppose $A \subseteq \emptyset$.
2. Since $\emptyset \subseteq A$ (Theorem 6.2.4), we have $(A \subseteq \emptyset) \wedge (\emptyset \subseteq A)$.
3. Hence, $A = \emptyset$ (by the equality of sets).

Alternative proof (proof by contradiction):

1. Suppose not, i.e., $A \subseteq \emptyset \wedge A \neq \emptyset$.
2. Since $A \neq \emptyset$, there exists an element $x \in A$.
3. This means $x \in \emptyset$ (as $A \subseteq \emptyset$), which contradicts the definition of an empty set.

(c) The statement is **true**. Proof:

$X \in \mathcal{P}(A \cap B)$	$\Leftrightarrow X \subseteq A \cap B$	(definition of power set)
	$\Leftrightarrow \forall x \in X \ (x \in A \cap B)$	(definition of subset)
	$\Leftrightarrow \forall x \in X \ (x \in A \wedge x \in B)$	(definition of set intersection)
	$\Leftrightarrow (\forall x \in X \ x \in A) \wedge (\forall x \in X \ x \in B)$	(definition of conjunction)
	$\Leftrightarrow (X \subseteq A) \wedge (X \subseteq B)$	(definition of subset)
	$\Leftrightarrow \big(X \in \mathcal{P}(A)\big) \wedge \big(X \in \mathcal{P}(B)\big)$	(definition of power set)
	$\Leftrightarrow X \in \mathcal{P}(A) \cap \mathcal{P}(B)$	(definition of set intersection)

Therefore, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ (by equality of sets).

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