CS1231S Assignment #2

AY2024/25 Semester 1

Deadline: Monday, 4 November 2024, 1:00pm

ANSWERS

IMPORTANT: Please read the instructions below carefully.

This is a graded assignment worth 10% of your final grade. There are **six questions** (an admin Q0 and six task questions Q1–6) with a total score of 40 marks. Please work on it by yourself, not in a group or discussion/in collaboration with anybody. Anyone found committing plagiarism (submitting other's work as your own), or sending your answers to others, or other forms of academic dishonesty, will be penalised with a straight zero for the assignment, and reported to the school. Please see SoC website "Preventing Plagiarism" https://www.comp.nus.edu.sg/cug/plagiarism/.

You must submit your assignment to **Canvas > Assignments > Assignment 2 submission** before the deadline. **Late submission will NOT be accepted**. We have set the closing time of submission to slightly after 1pm to give you a few minutes of grace, but in your mind, you should treat **1pm** as the deadline. If you think you might be too busy on the day of the deadline, please submit earlier. Also, avoid submitting in the last few minutes, as the system may get sluggish due to overload and you will miss the deadline, and we will not extend the deadline for you.

The question paper is on **Canvas > Files > Assignment 2**. In the same folder you can find the template submission files **assign2_template.docx** and **assign2_template.pdf**. Download one of these template files. You may proceed in one of these ways: type your answers in assign1_template.docx and convert it into a pdf file, or use a pdf editor and type your answers directly into assign1_template.pdf. You must rename the pdf file to **Axxxxxxxx.pdf** before you submit it, where Axxxxxxxx is your student number.

Note that we are using a software to scan your shaded Student Number, so the pdf file you submit must have the Student Number box (refer to the template file) at the exact position. Also, the answers you write must fall within the boxes at those fixed locations in the template file. So, **do not use your phone camera** to scan your file as it will likely not have the same pagination and positioning as the template files we provide.

You are to submit **a SINGLE pdf file**. You may submit multiple times, but only the last submitted file will be graded.

Note the following:

- Shade your **Student Number** correctly, as instructed on the template file.
- As this is an assignment given well ahead of time, we expect you to work on it early. You should submit **polished work**, not answers that are untidy or appear to have been done in a hurry, for example, with scribbling and cancellation all over the places. Marks may be deducted for untidy work.
- It is always good to be clear and leave no gap so that the marker does not need to make guesswork on your answer. When in doubt, the marker would usually not award the mark.
- Write/type your answers within the boxes given in the template file.
- Do not use any methods that have not been covered in class. When using a theorem or result that has appeared in class (lectures or tutorials), please quote the theorem number/name, the lecture and slide number, or the tutorial number and question number in that tutorial, failing which marks may be deducted. Remember to use numbering and give justification for important steps in your proof, or marks may be deducted.
- **[Very important!]** In the past, some students submitted a single-page file or even a blank file, or did not submit anything, without realising it! It is your responsibility to check that your submitted file is complete. We will not allow any submission after the deadline.

If you need any clarification about this assignment, please do NOT email us or post on telegram, but post on **QnA "Assignments"** topic so that everybody can read the answers to the queries.

Question 0. (Total: 2 marks)

Check that …

- you have submitted a pdf file with your Student Number as the filename. [1 mark]
- you have written both your name and tutorial group number (eg: T02) at the top of the first page of your file. (If you miss either one you will not be given any mark.) [1 mark]

Question 1. Counting and Probability (Total: 7 marks)

Show your workings. For example, instead of just writing the answer 720, write $5 \times 3! \times 4! = 720$, or instead of write 42, write $P(7,2) = 42$ to show your working.

(a) How many ways are there for 5 people to sit on a 3-seater sofa? There must be exactly 3 people on the sofa. [1 mark]

(b) How many ways are there for 5 people to sit on a 3-seater sofa, if two of them are friends who either both sit on the sofa or both do not sit? There must be exactly 3 people on the sofa.[2 marks]

(c) A box contains ten cards numbered from 1 to 10. Three cards are drawn at random one after the other. What is the probability, correct to three significant figures, that they are alternatively odd or even? That is, the cards drawn are odd, even, odd, or even, odd, even. [2 marks]

(d) If you roll a pair of fair n-sided dice one after another, where $n > 1$, what is the probability that the second die is higher than the first? Write your answer as a fraction. [2 marks]

Answers:

(a) $P(5,3) = \mathbf{120}$

(b) If the 2 friends don't sit, then there are $3! = 6$ ways.

If they sit, then there are $2!$ ways between them, and 3 ways for the remaining 3 persons, so $2! \times 3 = 6$ way.

Hence, altogether there are **12** ways.

(c) $2 \times \frac{5}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{5}{18} \approx 0.2778 \approx \mathbf{0.278}$

(d) P(two dice have the same number) = $\frac{n}{n^2} = \frac{1}{n}$. Hence, P(two dice have different numbers) = $\frac{n-1}{n}$. Probability of the second die higher than the first is the same as the probability of the first die higher than the second, by symmetry, hence, answer is $\boldsymbol{\frac{n-1}{2n}}$.

Question 2. Expected value (Total: 7 marks)

Show your workings.

(a) A drone making company found that one out of every 30 drones they make is faulty. The company makes a \$20 profit for each good drone, but suffers a loss of \$200 for each faulty drone. So, does the company make a profit or loss in the long term, and how much profit or loss do they make per drone? Write your answer correct to 4 significant digits. [2 marks]

(b) You draw a card from a standard deck of playing cards. If you pick a heart, you win \$10. If you pick a face card (Jack, Queen or King) which is not a heart, you win \$5. If you pick any other card, you lose \$5. Would you want to play the game? Explain your answer with the computed expected gain or loss. [2 marks]

(c) On average, how many times must a 6-sided fair die be rolled until a 3 turns up? [3 marks]

Answers:

(a) $E(X) = \frac{29}{30} \times 20 + \frac{1}{30} \times (-200) = \frac{380}{30} \approx 12.667 \approx 12.67.$

The company makes a profit of **\$12.67** per drone.

(b) Let X be the random variable that takes on the values 10, 5 and -5, the values of the winnings.

$P(X = 10) = \frac{13}{52}, P(X = 5) = \frac{9}{52}, P(X = -5) = \frac{30}{52}.$

$E(X) = P(X = 10) \times 10 + P(X = 5) \times 5 + P(X = -5) \times (-5) = \frac{130+45-150}{52} = \frac{25}{52} \approx 0.48$

You would play the game as the expected gain is **\$0.48** per game.

(c) On average, **6** times to roll the die until a 3 turns up.

One approach is to use geometric distribution:

Let X be the random variable representing the number of rolls until a 3 appears. Then $P(X = 1) = \frac{1}{6}; P(X = 2) = \frac{5}{6} \times \frac{1}{6}$, and so on. Hence, $P(X = k) = \left(\frac{5}{6}\right)^{k-1} \times \frac{1}{6}$. This is a geometric distribution with mean = 6.

That is, $\sum_{n=1}^{\infty} nP(X = n) = \sum_{n=1}^{\infty} n \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6} = \frac{6}{5} \times \frac{1}{6} \sum_{n=1}^{\infty} n \left(\frac{5}{6}\right)^{n} = \frac{1}{5} \times \frac{5/6}{(1-(5/6))^2} = 6$

Another (easier) approach:

The probability that a 3 appears is 1/6. If 3 doesn't appear (with probability 5/6), we have to start over. So, we have a 1/6 chance of rolling a 3 (and stopping), and a 5/6 chance of not rolling a 3, after which the number of rolls we expect to throw is the same as when we started. Hence,

$$E = \frac{1}{6} + \frac{5}{6}(E + 1)$$

Solving for E, we have $E = 6$.

Question 3. Functions (Total: 6 marks)

Let $A = \{a, b\}$ and S be the set of all strings over A. Define the function $f : S \to S$ as follows:

> $\forall s \in S$, $f(s)$ is a new string that replaces the leftmost occurrence of a in s with b. If a does not occur in s, then $f(s) = s$.

Examples: $f(\varepsilon) = \varepsilon$; $f(a) = b$; $f(bbb) = bbb$; $f(bbaab) = bbbab$.

Another function $g : S \to \mathbb{N}$ is defined as follows:

> $\forall s \in S$, $g(s) =$ (the number of a's in s) $-$ (the number of b's in s), or 0 if the difference is negative.

Examples: $g(\varepsilon) = 0$; $g(abaa) = 2$; $g(bb) = 0$; $g(babaa) = 1$; $g(abba) = 0$.

(a) What is $(f \circ f)(abaab)$? Show your workings. [1 mark]

(b) Is $f \circ f$ injective? Write "yes" or "no" followed by a proof or counterexample. [2 marks]

(c) Is g surjective? Write "yes" or "no" followed by a proof or counterexample. [3 marks]

Answers:

(a) $(f \circ f)(abaab) = f\big(f(abaab)\big) = f(bbaab) = \boldsymbol{bbbab}$.

(b) No. $(f \circ f)(aab) = f\big(f(aab)\big) = f(bab) = bbb = (f \circ f)(bbb)$ but $aab \neq bbb$.

(c) Yes.
Proof (by construction) (other constructions possible)

1. Take any $n \in \mathbb{N}$.
2. Let $s = aa \cdots a$ (exactly n a's). (Note: n is possibly 0. If so, $s = \varepsilon$.)
3. Hence, $s \in S$ since its members are all a's.
4. Now, $g(s) = n - 0 = n$ by the definition of g.
5. Therefore, g is surjective.

The trick is to construct a suitable $s \in S$ that works. Other constructions are possible, eg. a string containing $(n + k)$ a's followed by k b's.

Question 4. Hasse Diagrams (6 marks)

For each $n \in \mathbb{Z}^+$, let $D_n = \{d \in \mathbb{Z}^+ : d \mid n\}$. Figures 1, 2 and 3 are shown below.

Figure 1: D_a

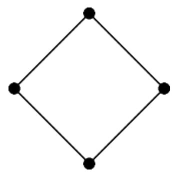


Figure 2: : D_b

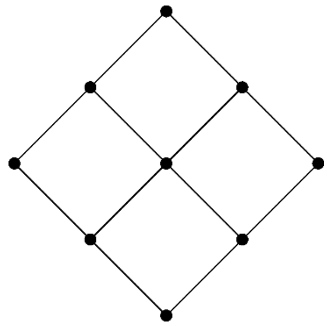


Figure 3: : D_c

The explanation you provide should be in text. Diagrams, if provided, are just for reference. Explanation using solely diagrams without explanation in text will not be accepted.

(a) Write out a value of a such that Figure 1 is a Hasse diagram for D_a with respect to the partial order "divides". Explain. [2 marks]

(b) Write out a value of b such that Figure 2 is a Hasse diagram for D_b with respect to the partial order "divides". Explain. [2 marks]

(c) Write out a value of c such that Figure 3 is a Hasse diagram for D_c with respect to the partial order "divides". Explain. [2 marks]

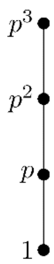
Answers:

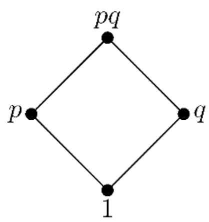
(a) The prime factorization of a must be of the form $\boldsymbol{p^3}$**, where $\boldsymbol{p}$ is a prime**.
Hence, a can be $2^3 = \mathbf{8}$, or $3^3 = \mathbf{27}$, or $5^3 = \mathbf{125}$, ….

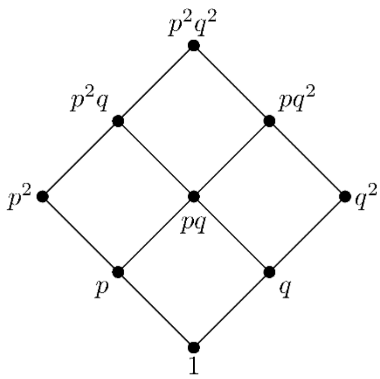
(b) The prime factorization of b must be of the form $\boldsymbol{pq}$**, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are primes**.
Hence, b can be $2 \times 3 = \mathbf{6}$, or $2 \times 5 = \mathbf{10}$, or $3 \times 5 = \mathbf{15}$, ….

(c) The prime factorization of c must be of the form $\boldsymbol{p^2q^2}$**, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are primes.**
Hence, c can be $2^2 3^2 = \mathbf{36}$, or $2^2 5^2 = \mathbf{100}$, or $3^2 5^2 = \mathbf{225}$, …

For reference:







Question 5. Mathematical Induction (6 marks)

Prove by induction that $2^{4n} - 2^n$ is divisible by 14 for all $n \in \mathbb{Z}_{\geqslant 0}$.

(Make sure you follow the structure of a mathematical induction proof given in class.)

Answer:

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition "$2^{4n} - 2^n$ is divisible by 14".
2. (Basis step) $P(0)$ is true because
$$2^{4\times 0} - 2^0 = 2^0 - 2^0 = 0 = 14 \times 0.$$
3. (Inductive step)
   - 3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(k)$ is true, i.e., that $2^{4k} - 2^k$ is divisible by 14.
   - 3.2. Use the definition of divisibility to find $\ell \in \mathbb{Z}$ satisfying $2^{4k} - 2^k = 14\ell$.
   - 3.3. Then $2^{4(k+1)} - 2^{k+1} = 2^{4k+4} - 2^{k+1}$
   - 3.4. $= 2^{4k} \times 2^4 - 2^{k+1}$
   - 3.5. $= (14\ell + 2^k) \times 2^4 - 2^{k+1}$ by the choice of ℓ;
   - 3.6. $= 14 \times 2^4\ell + 2^{k+4} - 2^{k+1}$
   - 3.7. $= 14 \times 2^4\ell + 2^{k+1}(2^3 - 1)$
   - 3.8. $= 14 \times 2^4\ell + 14 \times 2^k$
   - 3.9. $= 14(2^4\ell + 2^k)$, where $2^4\ell + 2^k \in \mathbb{Z}$ (by closure of integers under x and $+$)
   - 3.10. So $2^{4(k+1)} - 2^{k+1}$ is divisible by 14.
   - 3.11. Thus $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}$ $P(n)$ is true by MI.

Question 6. Function and Cardinality (Total: 6 marks)

Define $f: \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\} \to \mathbb{N}$ by setting $f(S)$ to be the smallest element of S whenever $S \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$. (Note that $\mathcal{P}(A)$ denotes the power set of A.)

For each of the following statements, state whether it is "true" or "false", followed by a proof or counterexample. Remember to quote an appropriate theorem/result to justify an important step.

If you are writing a proof, make sure it is no longer than eight lines, or it will not be graded.

(a) The function f has an inverse. [2 marks]

(b) $f^{-1}(\{n\})$ is uncountable for some $n \in \mathbb{N}$. [4 marks]

Answers:

(a) **False.**

$f(\{0\}) = 0 = f(\{0,1\})$, but $\{0\} \neq \{0,1\}$. So f is not injective, and hence not bijective. It follows that f does not have an inverse (by Theorem 7.2.3).

> Theorem 7.2.3:
> If $f: X \to Y$ is a bijection, then $f^{-1}: Y \to X$ is also a bijection.
> In other words, $f: X \to Y$ is bijective iff f has an inverse.

(b) **True.**

To prove $f^{-1}(\{0\})$ is uncountable.

1. Note that $f^{-1}(\{0\}) = \big\{\{0\} \cup S : S \in \mathcal{P}(\mathbb{Z}^+)\big\}$.
2. If S_1 and S_2 are different elements of $\mathcal{P}(\mathbb{Z}^+)$, then $\{0\} \cup S_1 \neq \{0\} \cup S_2$.
3. So $|f^{-1}(\{0\})| = |\mathcal{P}(\mathbb{Z}^+)|$.
4. As $\mathcal{P}(\mathbb{Z}^+)$ is uncountable (by tutorial 8 question 7), hence $f^{-1}(\{0\})$ is also uncountable.

> Tutorial 8 Question 7:
> Let A be a countably infinite set. Prove that $\mathcal{P}(A)$ is uncountable.

=== End of Paper ===