NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 2: AY2021/22)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY ONE (21)** questions in **THREE (3)** parts and comprises **TWELVE (12)** printed pages. The last two pages are intentionally left blank.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions and write your answers only on the **ANSWER SHEETS** provided.
5. Do **not** write your name on the ANSWER SHEETS.
6. The maximum mark of this assessment is 100.

Question	Max. mark
Part A: Q1 – 10	20
Part B: Q11 – 16	18
Part C: Q17	5
Part C: Q18	10
Part C: Q19	20
Part C: Q20	20
Part C: Q21	7
Total	**100**

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions [Total: 10×2 = 20 marks]

Each multiple choice question (MCQ) is worth **TWO marks** and has exactly **one** correct answer.

1. Which answer in this list is the correct answer to *this* question?

   A. All of the below.
   B. None of the below.
   C. All of the above.
   D. One of the above.
   E. None of the above.
   F. None of the above.

2. Aiken is taking the CS1231S final exam which consists of the dreadful MRQs (multiple-response questions). If each MRQ has 5 options A, B, C, D and E, in how many ways can Aiken write his answer for each MRQ, assuming that he writes at least one option in his answer. Note that the order of the options in his answer does not matter, for example, ABC, ACB and BCA are considered the same answer.

   A. 31.
   B. 32.
   C. 325.
   D. 3125.
   E. None of the above.

3. Which of the following statements is true?

   A. $(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$
   B. $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r)$
   C. $p \wedge (q \rightarrow r) \equiv (p \wedge q) \rightarrow (p \wedge r)$
   D. $p \vee (q \rightarrow r) \equiv (p \vee q) \rightarrow (p \vee r)$
   E. None of the above.

4. Given that $\forall x\, \exists y\, P(x, y)$ is true on a non-empty domain of discourse, which of the following statements is true?

   A. $\exists y\, \forall x\, P(x, y)$
   B. $\exists x\, \exists y\, P(x, y)$
   C. $\forall x\, \forall y\, P(x, y)$
   D. $\exists x\, \forall y \sim P(x, y)$
   E. None of the above.

5. Define a set S recursively as follows.

   (1) $5 \in S$. (base clause)

   (2) If $x \in S$, then $x + 3 \in S$ and $x + 5 \in S$. (recursion clause)

   (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

   What is the smallest integer k such that all integers $n \geq k$ are in S?

   A. 7.
   B. 8.
   C. 9.
   D. 10.
   E. None of the above.

6. Given the set $A = \{1,2,3,4,5\}$ and bijection $f: A \to A$ as follows:

$$f = \{(1,3), (2,4), (3,5), (4,1), (5,2)\}.$$

   What is the order of $f \circ f$?

   A. 2
   B. 3
   C. 4
   D. 5
   E. None of the above.

7. Let $S = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$, $V = \{\mathrm{A}, 2,3,4,5,6,7,8,9,10, \mathrm{J}, \mathrm{Q}, \mathrm{K}\}$,
   and $B = \{(\spadesuit, \mathrm{A}), (\heartsuit, 7), (\clubsuit, 9), (\spadesuit, 6), (\diamondsuit, \mathrm{J})\}$.

   Which of the following sets contains B as an element? (Note: $\mathcal{P}(X)$ denotes the power set of X).

   A. $S \times V$
   B. $S \cup V$
   C. $\mathcal{P}(S \times V)$
   D. $\mathcal{P}(S \cup V)$
   E. None of the above.

8. Given $A = \{1,2,3,4,5\}$ and the partial order R on A as follows:

$$R = \{(x, x) : x \in A\} \cup \{(2,1), (2,5), (3,1), (3,2), (3,4), (3,5), (4,1)\}$$

How many distinct linearizations of R are there?

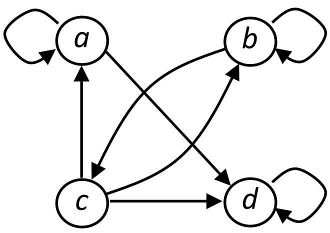
A. 3
B. 5
C. 7
D. 12
E. None of the above.

9. Aiken working on a problem from circuit design came up with a graph G, while Dueet working on a problem from computational biology came up with a graph G*. When they met for dinner, they were surprised to find that that G and G* are isomorphic. Aiken's graph G has 6 connected components.

How many connected components are there in G*?

A. G* has at most 6 connected components.
B. G* has at least 6 connected components.
C. G* has at exactly 6 connected components.
D. G* does not have exactly 6 connected components.
E. There is insufficient information to determine.

10. Given the following directed graph $G = (V, E)$, how many walks of length 3 in total are there between u and w, $\forall u, w \in V$?



A. 8.
B. 20.
C. 22.
D. 24.
E. 25.

Part B: Multiple Response Questions [Total: 6×3 = 18 marks]

Each multiple response question (MRQ) is worth **THREE marks** and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, B, C are the correct answers, write A, B, C.

Only if you get all the answers correct will you be awarded three marks. **No partial credit will be given for partially correct answers.**

11. Which of the following have the same answer?

    A. Number of subsets of {1,2,3,4,5,6,7,8,9,10} with cardinality 4.

    B. Number of ways to choose 6 out of 10 persons.

    C. Number of ways to arrange 6 persons around a round table.

    D. Number of solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 6$ where $x_i \in \mathbb{N}$.

    E. Number of permutations of ICANDOIT.

12. Given $A = \{1,2,3,4,5\}$ and the partial order R on A as follows:

    $$R = \{(x,x) : x \in A\} \cup \{(2,1),(2,5),(3,1),(3,2),(3,4),(3,5),(4,1)\}$$

    (This is the same partial order in Question 8.)

    Which of the following statements is/are true with respect to this partial order?

    A. 1 is a minimal element.

    B. 1 is a maximal element.

    C. 4 and 5 are non-comparable.

    D. 3 is a smallest element.

    E. 5 is a largest element.

13. Given the equivalence relation $\sim$ on $\mathbb{Z}^2$ defined by

    $$(a,b) \sim (c,d) \Leftrightarrow 3|(a-c) \wedge 2|(b-d).$$

    Which of the following is/are equivalence classes under this relation?

    A. $\{(x,y) \in \mathbb{Z}^2 : x, y \in \mathbb{Z}\}$

    B. $\{(x,y) \in \mathbb{Z}^2 : 3x, 2y \in \mathbb{Z}\}$

    C. $\{(x,y) \in \mathbb{Z}^2 : x = 3k-1, y = 2m \text{ where } k, m \in \mathbb{Z}\}$

    D. $\{(x,y) \in \mathbb{Z}^2 : x = 2k, y = 3m \text{ where } k, m \in \mathbb{Z}\}$

    E. $\{(x,y) \in \mathbb{Z}^2 : x = 3k+4, y = 2m+3 \text{ where } k, m \in \mathbb{Z}\}$

14. Suppose f and g are functions, which of the following statements is/are true?
    A. If f and g are injective, then $g \circ f$ is injective.
    B. If f and g are surjective, then $g \circ f$ is surjective.
    C. If f and g are bijective, then $g \circ f$ is bijective.
    D. If $g \circ f$ is bijective, then f and g are bijective.
    E. If $g \circ f$ is not bijective, then f and g are not bijective.

15. Which of the following sets is/are countable?
    A. The set A of all points in the plane with rational coordinates.
    B. The set B of all infinite sequences of integers.
    C. The set C of all functions $f: \{0,1\} \to \mathbb{N}$.
    D. The set D of all functions $f: \mathbb{N} \to \{0,1\}$.
    E. The set E of all 2-element subsets of $\mathbb{N}$.

16. The following are the pre-order traversal and post-order traversal of a binary tree:

    Pre-order: **U C N A D O I T !**

    Post-order: **N C D I ! T O A U**

    Which of the following is/are possible in-order traversals of this tree?
    A. C N U D A I O ! T
    B. C N U D A I O T !
    C. N C U D A I O T !
    D. N C U D A I ! O T
    E. None of the above.

Part C: There are 5 questions in this part [Total: 62 marks]

17. Prove by mathematical induction: $\mathbf{7} \mid (\mathbf{5^{2n+1}} + \mathbf{2^{2n+1}})$ for all $n \in \mathbb{N}$. **[Total: 5 marks]**

18. **[Total: 10 marks]**

(a) For each of the following functions, indicate whether it is injective and whether it is surjective. Write **True** or **False** on the Answer Sheets. You do not need to prove or disprove it. [6 marks]

(i) $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 1$.

(ii) $f: [3, \infty) \to [7, \infty)$ defined by $f(x) = (x - 3)^2 + 7$.

(iii) $f: \mathbb{Z}^+ \to \mathbb{Q}$ defined by $f(n) = 1/n$.

(b) Let $\preccurlyeq$ be a partial order on a set S. A subset C of S is called a **chain** if and only if each pair of elements in C is comparable, that is, $\forall a, b \in C\ (a \preccurlyeq b \vee b \preccurlyeq a)$. A **maximal chain** is a chain M such that $t \notin M \Rightarrow M \cup \{t\}$ is not a chain.

Given a finite set S with cardinality n, what is the maximum number of maximal chains that a partial order on S can have? Explain your answer. (Answer without explanation will not receive any credit.) [2 marks]

(c) Let $\preccurlyeq$ be a partial order on a set S. An **antichain** is a subset of S such that no two elements in it are comparable under $\preccurlyeq$. A **maximal antichain** is an antichain that is not a proper subset of any other antichain.

Let D_n be the set of positive divisors of integer n. Given a partial order on D_{30} under divisibility, write out all the maximal antichains in this partial order. [2 marks]

19. Counting and Probability **[Total: 20 marks]**

<u>Note that you need **not** show your working for parts (a) to (c)</u>.

(a) [Subtotal: 4 marks]




Three persons went to the famous Gluttons Gourmet Hawker Centre where four of the city's best hawker stalls are housed: DiDi Chicken Rice, Singing Char Kway Teow, Alamak Nasi Lemak and Hurry Hurry Curry Fish Head. Assume that the three of them are indistinguishable. If each of them is to order from one of these four stalls,

(i) in how many ways can they order their food? [1 mark]
(ii) in how many ways can they order their food from the same stall? [1 mark]
(ii) in how many ways can they order their food from two of the four stalls? [2 marks]

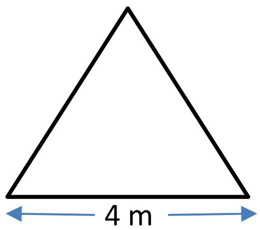
(b) [Subtotal: 5 marks]
A rare disease broke out in a city with a prevalence of 0.1%, that is, it affects 1 out of every 1000 persons. A quick test kit has been developed that has a sensitivity of 85%, which is the probability that a person with the rare disease is tested positive. Among those who took the test, 10% of the time it came out positive.

Write your answers correct to <u>3 significant figures</u>.

(i) Divoc has shown symptoms of the disease. Should he be tested positive, what is the probability that he actually has the disease? [2 marks]
(ii) What is the probability of a false positive result, that is, a person does not have the disease but is tested positive? [3 marks]

(c) [Subtotal: 6 marks]
How many equivalence relations are there on a set with n elements, where
(i) $n = 2$? [1 mark]
(ii) $n = 3$? [2 marks]
(iii) $n = 4$? [3 marks]

(d) [Subtotal: 5 marks]
(i) Dueet owns a beautiful cat called Meow who loves to wear socks on its four feet. Dueet has a collection of many cat socks in three colours: white, red and yellow. What is the least number of socks Dueet must pull out from the drawer to guarantee getting four socks of matching colour? You do not need to explain your answer. [2 marks]
(ii) Aiken has a mini garden which is an equilateral triangle with side of 4 metres. He wants to plant five seeds in his mini garden. Explain that there will be two seeds within 2 metres of each other. [3 marks]



20. Graphs and Trees [Total: 20 marks]

(a) Draw all non-isomorphic, connected simple graphs on four vertices. [3 marks]

(b) Prove or disprove the following statements: [8 marks]

(i) The graph $K_{2,4}$ is a planar graph.

(ii) The graph $K_{3,4}$ is a planar graph.

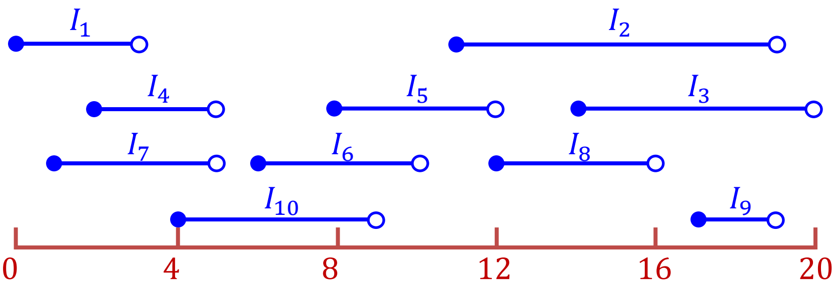
(iii) There exists a simple graph that is connected and planar with 5 vertices and 8 faces.

(iv) The graph $K_5 - \{e\}$ where e is any edge in the graph is 4-colourable.

(c) [Subtotal: 9 marks] You are given a set of n tasks $T = \{T_1, T_2, T_3, \ldots, T_n\}$. Each task T_k is represented by the interval $I_k = [s_k, e_k)$ where s_k is the *start time* and e_k (where $s_k < e_k$) is the *end time* of the task, for $k = 1,2,\ldots,n$. An instance of this problem with $n = 10$ is shown below.

Instance: $n = 10$, and $T = \{T_1, T_2, \ldots, T_{10}\}$, and

$I_1 = [0,3)$, $I_2 = [11,19)$, $I_3 = [14,20)$, $I_4 = [2,5)$, $I_5 = [8,12)$,
$I_6 = [6,10)$, $I_7 = [1,5)$, $I_8 = [12,16)$, $I_9 = [17,19)$, $I_{10} = [4,9)$.



Define a graph $G = (T, E)$ where

$$E = \{ \{T_x, T_y\} : (I_x \cap I_y \neq \emptyset) \}$$

Namely, there is an edge $\{T_x, T_y\} \in E$ if and only if the intervals $[s_x, e_x)$ of task T_x *overlaps* with the interval $[s_y, e_y)$ of task T_y. We call G an **interval graph**.

(i) Draw the interval graph G for this instance. Ten labelled vertices have been drawn on the Answer Sheets. [3 marks]

(ii) Give a minimum colouring for the graph G from (i) above. [2 marks]

(iii) You want to assign all the tasks to machines so as to *minimize* the number of machines used. Each machine can only work on one task at a time. Each machine can do any number of tasks as long as the tasks do not overlap in time. (You can assume that a machine can (if necessary) start on a new task immediately after finishing a previous task. In the instance above, tasks T_5 and T_8 can be assigned to the same machine, if necessary.)

Give a short argument (in one or two sentences) that the tasks assigned the same colour in the coloured graph G (obtained in part (ii) above) can be done by one machine. [2 marks]

(iv) Give an assignment of tasks in T to machines that minimizes the number of machines needed. You can use the one from part (c)(ii) above or a different assignment that minimizes the number of machines. [2 marks]

21. **[Total: 7 marks]**

(a) Define a partition of $\mathbb{Z}$ that divides $\mathbb{Z}$ into 1231 countably infinite subsets. You do not need to explain your answer [2 marks]

(b) Define a bijection $f: (0,1) \to (0,1]$ so that you may conclude that $(0,1)$ and $(0,1]$ have the same cardinality. Prove that your function f is a bijection. [5 marks]

=== END OF PAPER ===

This page is intentionally left blank.

This page is intentionally left blank.