NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 1: AY2022/23)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY-FIVE (25)** questions in **TWO (2)** parts and comprises **NINE (9)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions and write your answers only on the **ANSWER SHEET** provided.
5. Do **not** write your name on the ANSWER SHEET.
6. The maximum mark of this assessment is 100.

Question	Max. mark
Part A: Q1 – 20	40
Part B: Q21	4
Part B: Q22	20
Part B: Q23	20
Part B: Q24	10
Part B: Q25	6
Total	**100**

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions [Total: 20×2 = 40 marks]

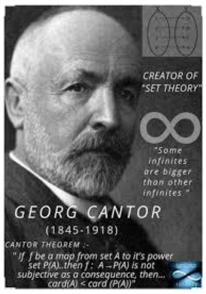
Each multiple choice question (MCQ) is worth **TWO marks** and has exactly **one** correct answer.

1. Which of the following is the hotel manager in the Infinite Hotel video shown in class?

A. 

B. 

C. 

D. 

2. Which of the following statements is logically equivalent to $\sim\boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r})$?
   - A. $(p \to q) \wedge (p \to r)$.
   - B. $\sim(p \wedge \sim q) \wedge (r \to p)$.
   - C. $(\sim p \wedge q) \vee (p \wedge \sim r)$.
   - D. $(p \wedge q) \to (\sim q \vee \sim r)$.
   - E. None of the above.

3. Which of the following statements is the negation of $\forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \forall \boldsymbol{z} \left(\left(\boldsymbol{P}(\boldsymbol{x}) \wedge \boldsymbol{Q}(\boldsymbol{y}) \right) \to \boldsymbol{R}(\boldsymbol{z}) \right)$?
   - A. $\forall x\ \exists y\ \forall z \sim \left(\left(P(x) \wedge Q(y) \right) \to R(z) \right)$.
   - B. $\exists x\ \forall y\ \exists z \left(\sim\left(P(x) \wedge Q(y) \right) \vee R(z) \right)$.
   - C. $\exists x\ \forall y\ \exists z \left(\sim R(z) \to \sim\left(P(x) \wedge Q(y) \right) \right)$.
   - D. $\exists x\ \forall y\ \exists z \left(P(x) \wedge Q(y) \wedge \sim R(z) \right)$.
   - E. None of the above.

4. Let $A = \{-2, -1, 0, 1, 2\}$, $B = \{0, 1, 4\}$ and $C = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. Which of the following statements are true?
   - (i) $\forall x \in C\ \left((x \in A) \leftrightarrow (x^2 \in B) \right)$.
   - (ii) $\forall x \in C\ \left((\forall y \in B\ xy \in B) \to (x^2 = x) \right)$.
   - (iii) $\exists x \in A\ \forall y \in A\ \left((x \neq 0) \wedge (xy \in B) \right)$.

   - A. Only (i).
   - B. Only (ii).
   - C. Only (i) and (ii).
   - D. Only (i) and (iii).
   - E. None of the above.

5. Let A and B be any sets, $A \cup B$ be the universal set, and let $\mathcal{P}(X)$ denote the power set of X. Which of the following statements are true?

   (i) If $A \cap B = \emptyset$, then $\mathcal{P}(A) \cap P(B) = \emptyset$.

   (ii) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

   (iii) $(A \cup \bar{B}) \cap (\bar{A} \cup B) = (A \cap \bar{B}) \cup (\bar{A} \cap B)$.

   A. Only (ii).
   B. Only (iii).
   C. Only (i) and (ii).
   D. Only (ii) and (iii).
   E. None of the above.

~~6. For any set A, suppose there exists a bijection $f: \mathcal{P}(A) \to \mathbb{Z}$, where $\mathcal{P}(A)$ denotes the power set of A, which of the following statements is true?~~

   ~~A. A is finite.~~
   ~~B. A is uncountably finite.~~
   ~~C. A is countably infinite.~~
   ~~D. A is uncountable.~~
   ~~E. None of the above.~~

7. Given the following statements on any set A,

   (i) If R is a reflexive relation on A, then $|A| \leq |R|$.

   (ii) If R is a symmetric relation on A, then $|A| \leq |R|$.

   (iii) If R is a transitive relation on A, then $|A| \leq |R|$.

   Which of the statements above are true?

   A. None of (i), (ii) or (iii) is true.
   B. Only (i) is true.
   C. Only (i) and (ii) are true.
   D. Only (ii) and (iii) are true.
   E. All of (i), (ii) and (iii) are true.

8. Consider the following relations R and S on $\mathbb{Z}_{\geq 2}$, where $a|b$ means a divides b.

$$xRy \Leftrightarrow \exists k \in \mathbb{Z}_{\geq 2}\ (k|x \wedge k|y)$$
$$xSy \Leftrightarrow \exists k \in \mathbb{Z}_{\geq 2}\ (x|k \wedge y|k)$$

   Which of the following statements is true?

   A. Neither R nor S is an equivalence relation.
   B. R is an equivalence relation but S is not.
   C. S is an equivalence relation but R is not.
   D. Both R and S are equivalence relations.

9. Given a set A with two elements, how many relations on A are not transitive?
   A. 1.
   B. 3.
   C. 8.
   D. 13.
   E. None of the above.

10. How many possible partitions are there on a set with 3 elements?
    A. 1.
    B. 3.
    C. 5.
    D. 7.
    E. None of the above.

11. Given $A = \{1,2,3,4,5\}$ and the partial order R on A as follows:
    $$R = \{(x,x) : x \in A\} \cup \{(4,1),(5,2),(4,2),(1,2),(4,5)\}.$$
    How many distinct linearizations of R are there?
    A. 2
    B. 7
    C. 8
    D. 10
    E. None of the above.

12. Given the partial order relation R on A in question 11 above, which of the following statements are true?
    (i) There is no largest element.
    (ii) The smallest element is 4.
    (iii) The maximal elements are 2 and 3.

    A. Only (i).
    B. Only (i) and (iii).
    C. Only (ii) and (iii).
    D. All of (i), (ii) and (iii).
    E. None of the above.

13. When flipped, a biased coin lands on its head with a probability of p, and lands on its tails with a probability of $(1-p)$. Dueet flips the coin ten times. What is the probability of getting exactly 3 heads, with all the 3 heads in a row, that is, the 3 heads appearing consecutively?
    - A. $3p^3(1-p)^7$.
    - B. $(3 \times 7)p^3(1-p)^7$.
    - C. $\binom{10}{3}p^3(1-p)^7$.
    - D. $8p^3(1-p)^7$.
    - E. None of the above.

14. Define a set S recursively as follows.
    - (1) $1 \in S$ and $2 \in S$. (base clause)
    - (2) If $x, y \in S$, then $\frac{x}{y} \in S$. (recursion clause)
    - (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

    Which of the following statements is true?
    - A. $\forall x \in S\ (x \in \mathbb{Z} \Rightarrow x$ is even).
    - B. $\forall x \in S\ (x \in \mathbb{Q} \Rightarrow x < 1)$.
    - C. $\forall x, y \in S\ (x + y \geq 1)$.
    - D. $|S| = |\mathbb{Z}|$.
    - E. None of the above.

15. Given the recurrence relation $a_n = 5n + a_{n-1}$ with initial value $a_0 = 4$, what is the value of a_{20}?
    - A. 1054
    - B. 1050
    - C. 1049
    - D. 1044
    - E. None of the above.

16. The **order** of a bijection $f: A \to A$ is defined to be the smallest $n \in \mathbb{Z}^+$ such that
$$\underbrace{f \circ f \circ \cdots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$
    Let set $A = \{1,2,3,4\}$ and the bijections $g: A \to A$ and $k: A \to A$ are given as follows:
$$g = \{(1,3), (2,4), (3,2), (4,1)\};$$
$$k = \{(1,3), (2,2), (3,1), (4,4)\}.$$
    What is the order of $g \circ k$?
    - A. 1
    - B. 2
    - C. 3
    - D. 4
    - E. None of the above.

17. Which of the following are countable sets?
    (i) The set of relations on $\mathbb{Z}$.
    (ii) The set of rational numbers between 0 and 1 exclusive.
    (iii) The set $\mathbb{Z}^*$ of all strings over $\mathbb{Z}$.

A. Only (i).
B. Only (ii).
C. Only (i) and (ii).
D. Only (ii) and (iii).
E. All of (i), (ii) and (iii).

18. Which of the following statements are true?
    (i) $K_5 - \{e\}$ is planar where e is one of the edges in K_5.
    (ii) K_5 is both Eulerian and Hamiltonian.
    (iii) There is a simple planar graph with 6 vertices, 8 edges, and 7 faces.

A. Only (i).
B. Only (i) and (ii).
C. Only (i) and (iii).
D. All of (i), (ii) and (iii).
E. None of the above.

19. Which of the following statements are true?
    (i) Let G be a simple, undirected graph with 6 vertices, 5 edges and no cycles. Then it is not possible to have only one vertex in G with degree 1.
    (ii) Let G be a simple, undirected graph with n vertices and e edges. If G is not connected, then $e < (n - 1)$.
    (iii) All cycles in the graph $K_{3,4}$ have an even number of edges.

A. Only (i).
B. Only (i) and (ii).
C. Only (i) and (iii).
D. All of (i), (ii) and (iii).
E. None of the above.

20. How many non-isomorphic simple, undirected, disconnected graphs with 4 vertices are there?

A. 2.
B. 3.
C. 4.
D. 5.
E. 6.

Part B: There are 5 questions in this part [Total: 60 marks]

21. [Total: 4 marks] Mathematical induction.

Let $f_0, f_1, f_2, \cdots$ be the Fibonacci sequence 0,1,1,2,3,5,8,13,21,…

Prove by mathematical induction that, for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} f_i^2 = f_n f_{n+1}.$$

Aiken has written the first line of his proof as shown below. Complete his proof.

1. For each $n \in \mathbb{N}$, let $P(n)$ be the proposition $\sum_{i=0}^{n} f_i^2 = f_n f_{n+1}$.

22. [Total: 20 marks] Counting and probability.

You do not need to show your working for this question.

(a) How many integer solutions for x_1, x_2 and x_3 does the following equation have, given that $x_i \geq i$, for $1 \leq i \leq 3$?

$$x_1 + x_2 + x_3 = 10.$$

Write your answer as a single number. [2 marks]

(b) Given two independent events A and B with $P(A) = 0.2$ and $P(A \cup B) = 0.6$, what is $P(B)$? Write your answer as a single number. [2 marks]

(c) Aiken rolled a fair 6-sided die twice. What is the probability that his second roll has a higher value than his first roll? Write your answer as a simple fraction. [2 marks]

(d) There are 838 students in CS1231S this semester. At the minimum how many students were born in the same month? [2 marks]

(e) A standard 52-card deck comprises 13 ranks in each of the four suits. The ranks are: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King. In this question, we assign a score to each rank: Ace is 1, Jack is 11, Queen is 12, King is 13 and the numerical cards have their numerical values as their scores.

In a certain game, the player picks 2 cards without replacement from a standard deck and calculates the combined score. For example, if the player picks a Queen and a 7, then the combined score is 19. What is the expected value of the combined score? [2 marks]

(f) You are writing a spam-detection software and have gathered some statistics: the probability of an email being a spam is 25%, the probability of the words "free gifts" appearing in a non-spam email is 30%, and the probability of the words "free gifts" appearing in a spam email is double that, which is 60%. Write your answers for the following parts as numerical values and not formulas.

(i) What is the probability of the words "free gifts" appearing in an email? [3 marks]

(ii) If one receives an email that contains the words "free gifts", what is the probability that the email is a spam? [2 marks]

(g) How many injective functions are there from set A to set B given the following:

(i) $|A| = 2$ and $|B| = 3$. Write your answer as a single integer. [2 marks]

(ii) $|A| = k$ and $|B| = n$, where $k \leq n$. Write your answer as a formula. [3 marks]

23. [Total: 20 marks] Graphs and trees.

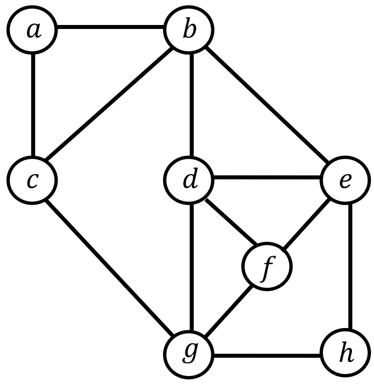
Definitions:

> Let $G = (V, E)$ be an undirected graph and v be a vertex in G. Then $G - v$ is the subgraph obtained by deleting the vertex v from V and deleting all edges incident to v from E.
>
> A vertex v is called a **cut-point** for G if $G - v$ is disconnected.
>
> Let $G = (V, E)$ be an undirected graph and e be an edge in G. Then $G - e$ is the subgraph obtained by deleting the edge e from E.
>
> An edge e is called a **bridge** for G if $G - e$ is disconnected.

Note: You do not need to explain or show your working for this question.

(a) Find an Eulerian trail in the undirected graph G shown on the right.

Label each edge traversed as 1,2,3,… in the order in which you traverse the Eulerian trail. For example, if your Eulerian trail is $x \to y \to z \to \cdots$, then label edge $\{x, y\}$ with 1, edge $\{y, z\}$ with 2, and so on. [2 marks]



(b) List out all the vertices where the Eulerian trail in part (a) may start. [2 marks]

(c) Is the following statement TRUE or FALSE?

"In a tree T with 2 or more vertices, every vertex of degree > 1 is a cut-point." [2 marks]

(d) Is the following statement TRUE or FALSE?

"In a tree T with 2 or more vertices, every edge is a bridge." [2 marks]

(e) The in-order and pre-order traversals of a binary tree T yield the following sequences of vertices. Give the post-order traversal. [2 marks]

In-order: E D B C G F A H K
Pre-order: C B D E A G F H K

For parts (f) to (i):

You are in charge of packing volatile items {A, B, C, D, E, F, G, H, J, K} into containers for shipping. Due to the volatile nature of the items, some items could be in conflict with other items. Two items are in conflict with each other means that they cannot be packed together or bad things might happen.

You are given the conflict list below, which states, for each item, those items that it conflicts with. For example, (from line 2) item B cannot be packed with A or C or D.

A: B, C;	**C:** A, B, D;	**E:** D, F, G, H;	**G:** E, H;	**J:** F, K;
B: A, C, D;	**D:** B, C, E, F;	**F:** D, E, J, K;	**H:** E, G;	**K:** F, J.

We want to pack these volatile items into containers so that no conflicting items are in the same container, and obviously, as computer scientists, we want to use a *minimum number of containers*. We call this the ***Volatile Item Packing Problem*** (***VIPP***).

(f) You want to model the **VIPP** as a graph colouring problem on a graph $G = (V, E)$. In this graph model, what do the vertices in G represent? When are two vertices adjacent in G? What do vertices of the same colour mean? [3 marks]

(g) Draw the conflict graph $G = (V, E)$ for the sample instance. For standardization, the vertices of the graph have already been drawn for you on the Answer Sheet. You just need to draw the edges. [3 marks]

(h) Colour G as in a graph colouring problem with the minimum number of colours. If you do not have enough pens of different colours, you may write 1, 2, 3, 4, … above each vertex, where each number represents a colour. [2 marks]

(i) Now, map your graph colouring solution in (h) to a solution of the original VIPP. Give the packing solution that corresponds to your coloured graph in (h). How many containers are needed in your packing solution? [2 marks]

24. [Total: 10 marks] Functions and relations.

(a) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be two bijections. The function $h: \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$h(x) = f(x) + g(x), \text{ for any } x \in \mathbb{R}.$$

Provide a counterexample to disprove this statement: h is a bijection. [2 marks]

(b) Definitions

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be **increasing** iff $x_2 > x_1 \to f(x_2) > f(x_1), \forall x_1, x_2 \in \mathbb{R}$.
A function $f: \mathbb{R} \to \mathbb{R}$ is said to be **decreasing** iff $x_2 > x_1 \to f(x_2) < f(x_1), \forall x_1, x_2 \in \mathbb{R}$.

Is the following statement true or false? If true, prove it; if false, disprove it.

If g is an increasing function and h is a decreasing function, then $h \circ g$ is a decreasing function. [3 marks]

(c) A **totally ordered set** is a pair $(A, \preccurlyeq)$ where A is a set and $\preccurlyeq$ is a total order on A.
Define a relation $\sim$ on two totally ordered sets as follows:

$(A, \preccurlyeq_A) \sim (B, \preccurlyeq_B)$ if and only if there exists a bijection $f: A \to B$ such that $\forall a_1, a_2 \in A, \quad a_1 \preccurlyeq_A a_2 \leftrightarrow f(a_1) \preccurlyeq_B f(a_2)$.

Prove that $(\mathbb{R}, \leq) \sim (\mathbb{R}, \geq)$, where $\leq$ and $\geq$ are the usual numerical less-than-or-equal-to and greater-than-or-equal-to symbols respectively. [5 marks]

25. [Total: 6 marks] Cardinality

Let A be a set such that $|A \times A| = |A|$.
Suppose there is a countably infinite subset B of A such that there is no surjection from B to A.

You may quote this without proof: Any subset of a finite set is finite.

(a) Is the statement "A is countable" true or false? Prove or disprove it. [2 marks]

(b) Suppose S is a countable subset of $A \times A$. Prove that there exists some element $x \in A$ such that $\{x\} \times A \subseteq (A \times A) \setminus S$. [4 marks]

=== END OF PAPER ===