NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 2: AY2022/23)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY-FIVE (25)** questions in **TWO (2)** parts and comprises **TWELVE (12)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions and write your answers only on the **ANSWER SHEET** provided.
5. Do **not** write your name on the ANSWER SHEET.
6. The maximum mark of this assessment is 100.

Question	Max. mark
Part A: Q1 – 20	40
Part B: Q21	5
Part B: Q22	20
Part B: Q23	20
Part B: Q24	9
Part B: Q25	6
Total	**100**

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions [Total: 20×2 = 40 marks]

Each multiple choice question (MCQ) is worth **TWO marks** and has exactly **one** correct answer.

1. Which of the following is a multiple-choice question in this semester's midterm test?
   - A. What are the venues where CS1231S lectures are held in this semester?
   - B. What is the date of this semester's midterm test?
   - C. What is the tagline of CS1231S?
   - D. What are the two names that are often mentioned in CS1231S?
   - E. None of the above.

2. Given the following statements:
   - (i) $\big((p \to q) \lor (r \to s)\big) \to (\sim p \lor r)$.
   - (ii) $\big((p \to q) \lor (r \to s)\big) \to \big((\sim q \to \sim p) \land (\sim s \to \sim r)\big)$.
   - (iii) $\big((p \lor q \to r) \land (p \lor q \to s)\big) \to \big((p \lor q) \to (r \lor s)\big)$.
   - (iv) $\big((p \lor q \to r) \land (p \lor q \to s)\big) \to \big((p \lor q) \to (r \land s)\big)$.

   Which of the above statements are tautologies?
   - A. Only (i), (iii) and (iv).
   - B. Only (ii), (iii) and (iv).
   - C. Only (iii) and (iv).
   - D. Only (iii).
   - E. None of the options (A), (B), (C), (D) are correct.

3. Let $A = \{-3,0,3\}$, $B = \{-3,-2,-1,0,1,2,3\}$ and $C = \{-4,-2,-1,0,1,2,4\}$. Given the following statements:
   - (i) $\forall x \in A\ \big((x \in B) \leftrightarrow (x - 1 \in C)\big)$.
   - (ii) $\exists x \in B\ \ \forall y \in A\ \ \forall z \in C\ (xy \le z)$.
   - (iii) $\forall x \in C\ \ \exists y \in B\ \ \forall z \in A\ (x - y < z)$.

   Which of the above statements are true?
   - A. Only (i).
   - B. Only (ii).
   - C. Only (i) and (ii).
   - D. Only (i) and (iii).
   - E. None of the options (A), (B), (C), (D) are correct.

4. Two sets A and B are defined as follows:

$$A = \{n \in \mathbb{Z}^+ : (n > 1) \land \forall r, s \in \mathbb{Z}^+ \left(n = rs \rightarrow (r = n) \lor (r = 1)\right)\};$$
$$B = \{n \in \mathbb{Z}^+ : \exists r, s \in \mathbb{Z}^+ \left(n = rs \land (1 < r < n) \land (1 < s < n)\right)\}.$$

Which of the following are true?

(i) $A \cap B = \emptyset$.
(ii) $A \cup B = \mathbb{Z}^+$.
(iii) 6 is the smallest value in B.

A. Only (i).
B. Only (ii).
C. Only (i) and (ii).
D. Only (i) and (iii).
E. All of (i), (ii) and (iii).

5. Referring to the sets A and B in question 4 above, which of the following are true? (Note that $\mathcal{P}(X)$ is the power set of X.)

(i) A and B are countable.
(ii) A is countable but B is uncountable.
(iii) $\mathcal{P}(A)$ is countable.

A. Only (i).
B. Only (ii).
C. Only (i) and (iii).
D. Only (ii) and (iii).
E. None of the options (A), (B), (C), (D) are correct.

6. How many partitions of a set with 4 elements are there?

A. 12
B. 15
C. 16
D. 18
E. None of the above.

The following definitions are referred to by questions 7 to 9.

Given a function $f : A \to B$, we say that

- $g : B \to A$ is a **left inverse** of f if and only if $g\big(f(a)\big) = a$ for all $a \in A$.
- $h : B \to A$ is a **right inverse** of f if and only if $f\big(h(b)\big) = b$ for all $b \in B$.

The following functions are referred to by questions 7 and 8. $Bool$ is the set {**true**, **false**}.

(i) $f_1 : \mathbb{Q} \to \mathbb{Q}$;
$x \mapsto 12x + 31$.

(ii) $f_2 : Bool^2 \to Bool$;
$(p, q) \mapsto p \wedge {\sim}q$.

(iii) $f_3 : Bool^2 \to Bool^2$;
$(p, q) \mapsto (p \wedge q, p \vee q)$.

(iv) $f_4 : \mathbb{Z} \to \mathbb{Z}$;
$x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$

7. Which of the following functions have a left inverse?
   A. Only f_1.
   B. Only f_2.
   C. Only f_1 and f_2.
   D. Only f_3 and f_4.
   E. None of the options (A), (B), (C), (D) are correct.

8. Which of the following functions have a right inverse?
   A. Only f_1.
   B. Only f_2.
   C. Only f_1 and f_2.
   D. Only f_3 and f_4.
   E. None of the options (A), (B), (C), (D) are correct.

9. Which of the following statements are true?
   (i) If a function has a left inverse, then it has a right inverse.
   (ii) If a function has a right inverse, then it has a left inverse.
   A. Only (i) is true.
   B. Only (ii) is true.
   C. Both (i) and (ii) are true.
   D. Neither (i) nor (ii) is true.

10. Given the following three sets S_1, S_2, S_3 that are defined recursively:

S_1: (1) $1 \in S_1$. (base clause)
(2) If $n \in S_1$, then $2n - 1 \in S_1$. (recursion clause)
(3) Membership for S_1 can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

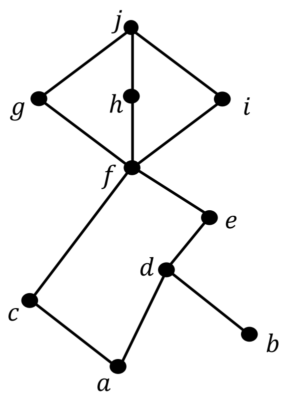
S_2: (1) $1 \in S_2$. (base clause)
(2) If $n \in S_2$, then $2n + 1 \in S_2$. (recursion clause)
(3) Membership for S_2 can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

S_3: (1) $1 \in S_3$. (base clause)
(2) If $n \in S_3$, then $2n - 1 \in S_3$ and $2n + 1 \in S_3$. (recursion clause)
(3) Membership for S_3 can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the above sets are the set of positive odd integers?

A. Only S_2.
B. Only S_3.
C. Only S_1 and S_3.
D. Only S_2 and S_3.
E. None of the options (A), (B), (C), (D) are correct.

11. The Hasse diagram of a partial order $\preccurlyeq$ on a set $K = \{a, b, c, d, e, f, g, h, i, j\}$ is shown below.



How many possible linearization of $\preccurlyeq$ are there?

A. 36
B. 42
C. 48
D. 54
E. None of the above.

For questions 12 to 15, refer to the definitions below:

Let $S \subseteq A$ in the poset $(A, \preccurlyeq)$.

Upper Bound:
If there exists an element $u \in A$ such that $s \preccurlyeq u$ for all $s \in S$, then u is called an *upper bound* of S.

Lower Bound:
If there exists an element $l \in A$ such that $l \preccurlyeq s$ for all $s \in S$, then l is called a *lower bound* of S.

Least Upper Bound:
If a is an upper bound of S such that $a \preccurlyeq u$ for all upper bound u of S, then a is the *least upper bound* of S, denoted by $lub(S)$.

Greatest Lower Bound:
If a is a lower bound of S such that $l \preccurlyeq a$ for all lower bound l of S, then a is the *greatest lower bound* of S, denoted by $glb(S)$.

Lattice:
A poset in which every pair of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

12. Given the poset $(\mathbb{Z}^+ \setminus \mathbb{Z}_{>120}, |)$ where | is the "divides" relation, which of the following are upper bounds of {2,3,4,5}?
    (i) 60
    (ii) 90
    (iii) 120

    A. Only (i).
    B. Only (iii).
    C. Only (i) and (ii).
    D. Only (i) and (iii).
    E. None of the options (A), (B), (C), (D) are correct.

13. Given the poset $(\mathcal{P}(\{a, b, c, d\}), \subseteq)$, which of the following are lower bounds of $\{\{b, c\}, \{c, d\}\}$?
    (i) $\emptyset$
    (ii) $\{c\}$
    (iii) $\{b, c, d\}$

    A. Only (i).
    B. Only (ii).
    C. Only (i) and (ii).
    D. Only (i) and (iii).
    E. None of the options (A), (B), (C), (D) are correct.

14. Given the poset $(\mathcal{P}(\{a,b,c,d\}),\subseteq)$, which of the following is the least upper bound of $\{\{b,c\},\{c,d\}\}$?

    A. $\emptyset$.

    B. $\{c\}$.

    C. $\{b,d\}$.

    D. $\{b,c,d\}$.

    E. $\{a,b,c,d\}$.

15. Which of the following are lattices?

    (i) The poset $(\mathbb{Z}^+ \setminus \mathbb{Z}_{>120}, |)$.

    (ii) The poset $(\mathcal{P}(\{a,b,c,d\}),\subseteq)$.

    (iii) The poset $(K, \preccurlyeq)$ in question 11.

    A. Only (i).

    B. Only (ii).

    C. Only (i) and (ii).

    D. Only (i) and (iii).

    E. None of the options (A), (B), (C), (D) are correct.

16. Given any non-empty finite set A, let $\mathcal{P}(A)$ be the power set of A and $\mathcal{C}$ any partition of A. Which of the following statements are always true?

    (i) $\exists S \in \mathcal{C}$ such that $S \subseteq \mathcal{P}(A)$.

    (ii) $\exists S \in \mathcal{P}(A)$ such that $S \in \mathcal{C}$.

    (iii) $\exists S \in A$ such that $\{S\} \in \mathcal{C}$.

    A. Only (i).

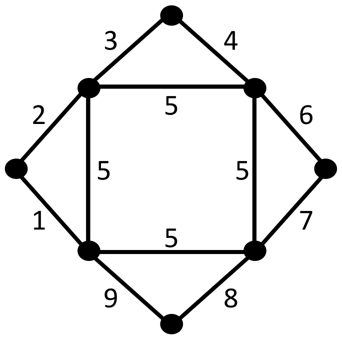
    B. Only (ii).

    C. Only (i) and (ii).

    D. Only (i) and (iii).

    E. None of the options (A), (B), (C), (D) are correct.

17. Given the following weighted graph, what is the weight of its minimum spanning tree?



A. 27
B. 28
C. 29
D. 31
E. None of the above.

18. Let G be a non-planar connected simple graph with the least possible number of edges. Which of the following statements on G is correct?

A. G has 10 edges and 6 vertices.
B. G has 10 edges and 5 vertices.
C. G has 9 edges and 6 vertices.
D. G has 9 edges and 5 vertices.
E. None of the above.

19. How many simple undirected graphs (not necessarily connected) can be constructed out of a vertex set of n vertices?

A. $\frac{n(n-1)}{2}$
B. 2^n
C. $\frac{n(n+1)}{2}$
D. $2^{\frac{n(n-1)}{2}}$
E. None of the above.

20. A **height-balanced binary tree** is a binary tree in which the height of the left subtree and the height of the right subtree of any vertex differ by at most one. (By convention, if a vertex does not have a left/right child, then the height of the absent left/right subtree is -1.)

    What is the minimum number of vertices in a height-balanced binary tree of height 4?

    A. 7.

    B. 9.

    C. 12.

    D. 23.

    E. None of the above.

Part B: There are 5 questions in this part [Total: 60 marks]

21. [Total: 5 marks] Mathematical induction.

The Fibonacci sequence F_n is defined for $n \in \mathbb{N}$ as follows:

$$F_0 = 0, \qquad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for } n > 1.$$

Prove by Mathematical Induction the following statement:

$$\boldsymbol{Even(F_n) \Leftrightarrow Even(F_{n+3}), \forall n \in \mathbb{N}.}$$

The predicate $Even(x)$ is true when the integer x is even and false otherwise. You may use the following fact in your proof:

Fact 1: $Even(x + y) \Leftrightarrow \big(Even(x) \Leftrightarrow Even(y)\big)$

(No marks will be awarded if you do not prove by mathematical induction.)

22. [Total: 20 marks] Graphs and trees.

Refer to the definitions below:

> **Cycle Graph:**
> A cycle graph C_n , $n \geq 3$, is an undirected graph consisting of n vertices $v_1, v_2, \ldots, v_n$ and n edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.
>
> **Star Graph:**
> A star graph S_n , $n \geq 3$, is an undirected graph consisting of $n + 1$ vertices $v_1, v_2, \ldots, v_n, v_{n+1}$ and n edges $\{v_1, v_{n+1}\}, \{v_2, v_{n+1}\}, \ldots, \{v_{n-1}, v_{n+1}\}, \{v_n, v_{n+1}\}$.
>
> **Wheel Graph:**
> A wheel graph W_n , $n \geq 3$, is defined as follows: the vertex set of W_n is the vertex set of S_n and the edge set of W_n is the union of the edge set of C_n and the edge set of S_n.

(a) Which of the following are complete graphs: C_4, S_4, W_4? [1 mark]

(b) Which of the following are planar graphs: $C_{1231}, S_{1231}, W_{1231}$? [1 mark]

(c) Does C_{1231} have an Euler circuit? Does W_{1231} have an Euler circuit? [2 marks]

(d) Determine the minimum number of colours needed to colour C_5, S_5 and W_5 such that no two adjacent vertices have the same colour. [3 marks]

(e) Draw all non-isomorphic spanning trees of C_5, S_5, W_5. [4 marks]

(f) Hamiltonian circuit is a closed-loop, i.e., it starts and ends at the same vertex while visiting every other vertex in the graph at most once. On the other hand, a Hamiltonian path is an open-loop and it visits each vertex only once. In summary, Hamiltonian path is an open-loop Hamiltonian circuit with different start and end vertices and must visit each vertex only once. Draw the possibility tree to determine the number of Hamiltonian paths in W_4, assuming that the initial vertex is fixed as v_1. [2 marks]

(g) Find the number of unique open-loop Hamiltonian paths in W_4. [1 mark]

(h) The adjacent matrix A of a directed graph is given below, where $a_{i,j}$ represents the number of edges from v_i to v_j for $i, j \in \{1,2,3,4\}$.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Find the number of walks of length 2 from to v_4 to v_2, and the number of walks of length 3 from v_2 to v_4. [2 marks]

(i) The post-order traversal and in-order traversal of a binary tree are given below. What is the height of the binary tree? [2 marks]

Post-order: 8, 9, 6, 7, 4, 5, 2, 3, 1.
In-order: 8, 6, 9, 4, 7, 2, 5, 1, 3.

(j) Consider an undirected graph G whose connected components are $H_1, H_2, H_3, \cdots, H_k$ where $k > 1$. Suppose $G = (V, E)$ and $H_i = (V_i, E_i)$ for $1 \leq i \leq k$. Is $\{V_1, V_2, \cdots, V_k\}$ always a partition of V? Is $\{E_1, E_2, \cdots, E_k\}$ always a partition of E? [2 marks]

23. [Total: 20 marks] Counting and probability.

You do not need to show your working for this question.

(a) Aiken has thrown n fair 6-sided dice simultaneously, where $n > 0$. What is the probability that all n dice have the same number of dots on the faces showing up? [1 mark]

(b) NUS School of Computing offers 129 modules and there are 39 time slots during which classes can be scheduled. At least how many lecture theatres will be needed with the assumption that each module is taught in one lecture theatre in one time slot? [2 marks]

(c) How many ways can 2 boys and 8 girls sit around a circle such that the 2 boys are not seated together? Write your answer as a value and not a formula. [2 marks]

(d) Assume that 8% of all athletes cheat via the usage of performance-enhancing drugs. A drug-cheating athlete tests positive for the performance-enhancing drug 96% of the time. On the other hand, those who do not cheat also test positive for performance-enhancing drug 9% of the time. What is the probability that a randomly selected athlete who tests positive for performance-enhancing drug is actually a drug-cheating athlete? Write your answer as a numerical value correct to 3 significant figures and not a formula. [2 marks]

(e) [Total: 5 marks]
Definition: A **Probability Mass Function (PMF)** is a function over the sample space $\mathcal{S}$ of a discrete random variable X which gives the probability that X is equal to a certain value. Let the function $f : \mathcal{S} \to \mathbb{R}$ be the PMF. The function f satisfies the following two properties:

P1: $f(k) = P(X = k)$

P2: $\sum_{k \in \mathcal{S}} f(k) = 1$

where $P(X = k)$ is the probability that the random variable X takes the value k.

Let the sample space $\mathcal{S}$ of a discrete random variable X be the set of integers $\{1,2,3,4,5,6\}$. The PMF of a random variable X is given as $f(k) = c(2k + 1)$ where c is a constant.

(i) Find the value of c. Write your answer as a simple fraction. [1 mark]

(ii) Find $P(X = 2)$. Write your answer as a simple fraction. [1 mark]

(iii) Find the expected value of X. Write your answer as a simple fraction or a value correct to 4 significant figures. [3 marks]

(f) [Total: 8 marks]
Let $A = \{a, b, c, \cdots, z\}$ be the set of lower-case letters and S be the set of all strings of length 6 over A. Five of the 26 letters – a, e, i, o and u – are vowels and the rest are consonants. Note that the letters in a string need not be distinct. Write your answers for the following parts as integers instead of formulas.

(i) Let $S_{1v} \subseteq S$ where each element in S_{1v} contains exactly one vowel. What is $|S_{1v}|$? [2 marks]

(ii) Let $S_{2v} \subseteq S$ where each element in S_{2v} contains exactly two vowels. What is $|S_{2v}|$? [2 marks]

(iii) Let $S_{\geq 1v} \subseteq S$ where each element in S_{1v} contains at least one vowel. What is $|S_{\geq 1v}|$? [2 marks]

(iv) Let $S_{\geq 2v} \subseteq S$ where each element in S_{2v} contains at least two vowels. What is $|S_{\geq 2v}|$? [2 marks]

24. [Total: 9 marks] Relations and functions.

(a) Let $\sim$ be an equivalence relation on X and let $g : X \rightarrow Y$ be a function such that

$$g(a) = g(b) \Leftrightarrow a \sim b \quad \forall a, b \in X.$$

Prove or disprove the following statement:

The following function f is well-defined:

$f : X/\sim \rightarrow Y$ given by the formula $f([x]) = g(x)\ \forall x \in X$. [4 marks]

(b) If the function f in part (a) above is well-defined, prove or disprove whether function f is injective or not injective.

If the function f in part (a) above is not well-defined, would changing the function g in part (a) to

$g(a) = g(b) \Leftrightarrow a \nsim b \quad \forall a, b \in X$. (Note: $\nsim$ is the negation of $\sim$.)

make the function f in part (a) well-defined? Prove or disprove.

(Part (b) will be graded only if you answered part (a) correctly.) [2 marks]

(c) Given the following bijections f and g on the set $A = \{1,2,3,4\}$,

$$f = \{(1,2), (2,4), (3,1), (4,3)\};$$

$$g = \{(1,4), (2,1), (3,2), (4,3)\}.$$

Find the order of f, g and $(f^{-1} \circ g)$. You do not need to show your working. [3 marks]

25. [Total: 6 marks] Cardinality.

Given the following sets:

$$A = \{(a_0, a_1, a_2, \cdots) : \forall i \in \mathbb{N}\ (a_i \in \{0,1\})\}.$$

$$B = \{(a_0, a_1, a_2, \cdots) \in A : \exists k \in \mathbb{Z}^+\ \forall n \geq k\ (a_n = 0)\}.$$

Prove or disprove: B is countable.

(Hint: Recall that for each positive integer n, there exists a **unique** binary representation given by $n = \sum_{i=0}^{m} a_i 2^i$ for some non-negative integer m and $(m+1)$-tuple $(a_0, a_1, \cdots, a_m) \in \{0,1\}^{m+1}$ such that $a_m = 1$.)

=== END OF PAPER ===