NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 1: AY2023/24)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY SIX (26)** questions in **TWO (2)** parts and comprises **ELEVEN (11)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions and write your answers only on the **ANSWER SHEETS** provided.
5. Do **not** write your name on the ANSWER SHEETS.
6. The maximum mark of this assessment is 100.

Question	Max. mark
Part A: Q1 – 22	44
Part B: Q23	6
Part B: Q24	20
Part B: Q25	20
Part B: Q26	10
Total	**100**

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions [Total: 22×2 = 44 marks]

Each multiple choice question (MCQ) is worth **TWO marks** and has exactly **one** correct answer.

1. Which of the following is not covered in CS1231S this semester?

    A. Folkman's Theorem.
    B. The Handshake Theorem in Graph Theory.
    C. The Generalized Pigeonhole Principle.
    D. Cantor's Diagonalization Argument.
    E. None of the above.

2. The island of Wantuutrewan is inhabited by exactly two types of natives: **knights** who always tell the truth and **knaves** who always lie. Every native is a knight or a knave, but not both.

    You meet two islanders *A* and *B* and this is what *A* says: "I am a knave and *B* is my type."

    Which of the following is true?

    A. Both *A* and *B* are knaves.
    B. *A* is a knight and *B* is a knave.
    C. *A* is a knave and *B* is a knight.
    D. Both *A* and *B* are knights.
    E. None of the above.

3. Define the logical connective ↓ as follows: $p \downarrow q \equiv \sim(p \vee q)$.
    Which of the following is logically equivalent to

    $$\big((p \downarrow q) \downarrow (r \downarrow s)\big) \rightarrow p\ ?$$

    A. p
    B. $\sim p \vee (\sim r \wedge \sim s)$
    C. $p \vee \sim q \vee (\sim r \wedge \sim s)$
    D. $true$
    E. None of the above.

4. Consider a propositional statement involving three variables p, q and r:

    $$\big(q \wedge (p \rightarrow r)\big) \rightarrow r$$

    In tabulating a truth table to validate the statement, which of the following is NOT a critical row to help validate this statement?

    A. $(p \equiv false) \wedge (q \equiv true) \wedge (r \equiv false)$.
    B. $(p \equiv false) \wedge (q \equiv true) \wedge (r \equiv true)$.
    C. $(p \equiv true) \wedge (q \equiv true) \wedge (r \equiv false)$.
    D. $(p \equiv true) \wedge (q \equiv true) \wedge (r \equiv true)$.
    E. None of the above.

5. Consider the predicate $P(x, y, z) \equiv$ "$xyz = 1$" for $x, y, z \in \mathbb{R}^+$. Which of the following statements are true on the domain $\mathbb{R}^+$?

   (i) $\forall x\ \forall y\ \exists z\ P(x, y, z)$.
   (ii) $\forall x\ \forall y\ \forall z\ P(x, y, z)$.
   (iii) $\exists x\ \forall y\ \forall z\ P(x, y, z)$.

   A. Only (i).
   B. Only (ii).
   C. Only (iii).
   D. Only (ii) and (iii).
   E. None of options (A), (B), (C), (D) is correct.

6. Consider the following set builder expression:

   $$\{x \in Y : Y \in B \wedge isSet(Y)\}$$

   where $isSet(Y)$ returns true if and only if Y is a set.

   What is returned when B is $\left\{ \{a\},\ \{\{a\}, b\},\ \emptyset,\ \{c\} \right\}$?

   A. $\{a, b, c\}$
   B. $\{a, b, c, \{a\}\}$
   C. $\{b, c, \{a\}\}$
   D. $\left\{ \emptyset, \{a\}, \{c\}, \{\{a\}, b\} \right\}$
   E. None of the above.

7. What is $|\ \{(x, y) \in A \cup B : 2 \leq x \leq 5 \text{ and } 3 \leq y \leq 7\}\ |$, where

   $A = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x = 2z \text{ for some } z \in \mathbb{Z}\}$ and
   $B = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y = 3z \text{ for some } z \in \mathbb{Z}\}$?

   A. 4.
   B. 12.
   C. 14.
   D. 18.
   E. None of the above.

8. Define the following relation on $A = \{1,2,3\}$:

   $$R = \{\ (1,1), (1,2), (2,1), (2,2), (3,3)\ \}.$$

   Which of the following is the relation $R \circ R \circ R \circ R \circ R \circ R$?

   A. $\{\ (1,1), (2,2)\ \}$.
   B. $\{\ (1,1), (2,2), (3,3)\ \}$.
   C. $\{\ (1,1), (1,2), (2,1), (2,2)\ \}$.
   D. $\{\ (1,1), (1,2), (2,1), (2,2), (3,3)\ \}$.
   E. None of the above.

9. Recall that $[a, b]$ denotes the set of all real numbers x such that $a \leq x \leq b$. Which of the following could be the set of equivalence classes $[0, 1]/R$ for some equivalence relation R?

   A. $[0,1]$.
   B. $\{\{x\} : x \in [0,1]\}$.
   C. $\mathcal{P}([0,1])$.
   D. $\{[x, x + 0.1] : x \in \{0, 0.1, 0.2, \ldots, 0.9\}\}$.
   E. None of the above.

10. Let R be a partial order on $\mathbb{Z} \times \mathbb{Z}$ such that

$$\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \left((a, b)\ R\ (c, d) \Leftrightarrow (a \leq c) \wedge (b \leq d)\right).$$

Consider relations S, T, U, V on $\mathbb{Z} \times \mathbb{Z}$ defined below:

   (i) Relation S: $\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \left((a, b)\ S\ (c, d) \Leftrightarrow a + b \leq c + d\right)$.
   (ii) Relation T: $\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \left((a, b)\ T\ (c, d) \Leftrightarrow a - b \leq c - d\right)$.
   (iii) Relation U: $\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \left((a, b)\ U\ (c, d) \Leftrightarrow ab \leq cd\right)$.
   (iv) Relation V: $\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \left((a, b)\ V\ (c, d) \Leftrightarrow a/b \leq c/d\right)$.

Which of the relations S, T, U, V are possible linearizations of R?

   A. Only S.
   B. Only S and T.
   C. Only S and U.
   D. Only S, T and U.
   E. None of options (A), (B), (C), (D) is correct.

11. Let A and B be sets. Let R be a relation from A to B such that

$$\forall (x, y) \in A \times B\ (x\ R\ y \Leftrightarrow y = x + 1).$$

For which of the following pairs of A and B, is R a function?

   (i) $A = B = \emptyset$.
   (ii) $A = \{1\}$ and $B = \{0\}$.
   (iii) $A = \mathbb{Z}$ and $B = \{0\}$.
   (iv) $A = \{1\}$ and $B = \mathbb{Z}$.

   A. Only (i).
   B. Only (i) and (iv).
   C. Only (ii) and (iii).
   D. Only (ii) and (iv).
   E. None of options (A), (B), (C), (D) is correct.

12. Which of the following are true?

    (i) If $f : A \to B$ is injective, then there is a function $g : C \to A$ such that $f \circ g$ is bijective.
    (ii) If $f : A \to B$ is injective, then there is a function $g : B \to C$ such that $g \circ f$ is bijective.
    (iii) If $f : A \to B$ is surjective, then there is a function $g : C \to A$ such that $f \circ g$ is bijective.
    (iv) If $f : A \to B$ is surjective, then there is a function $g : B \to C$ such that $g \circ f$ is bijective.

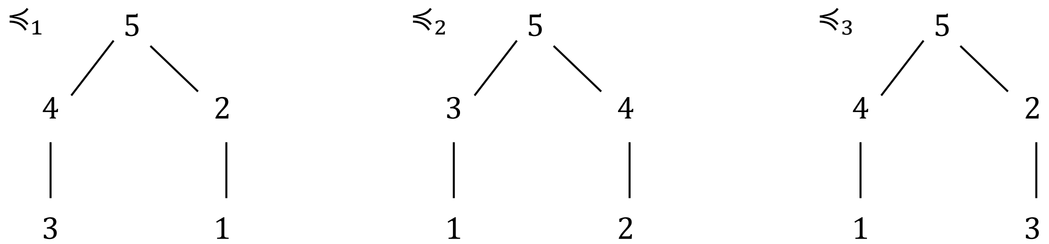
    A. Only (i) and (ii).
    B. Only (i) and (iv).
    C. Only (ii) and (iii).
    D. Only (iii) and (iv).
    E. None of options (A), (B), (C), (D) is correct.

13. Let $f : A \to A$ be a bijection. Which of the following are true?

    (i) If the order of $f \circ f$ is 1, then the order of f is 2.
    (ii) If the order of $f \circ f$ is 2, then the order of f is 4.
    (iii) If the order of f is 2, then the order of $f \circ f$ is 1.
    (iv) If the order of f is 4, then the order of $f \circ f$ is 2.

    A. Only (i) and (ii).
    B. Only (iii) and (iv).
    C. Only (i), (ii) and (iii).
    D. Only (ii), (iii) and (iv).
    E. None of options (A), (B), (C), (D) is correct.

14. The following are respectively Hasse diagrams of the partial orders $\preccurlyeq_1$, $\preccurlyeq_2$, $\preccurlyeq_3$ on $\{1,2,3,4,5\}$:



    Which of the following is true?

    A. There is a total order on $\{1,2,3,4,5\}$ that is a linearization of only both $\preccurlyeq_1$ and $\preccurlyeq_2$.
    B. There is a total order on $\{1,2,3,4,5\}$ that is a linearization of only both $\preccurlyeq_2$ and $\preccurlyeq_3$.
    C. There is a total order on $\{1,2,3,4,5\}$ that is a linearization of only both $\preccurlyeq_1$ and $\preccurlyeq_3$.
    D. There is a total order on $\{1,2,3,4,5\}$ that is a linearization of all of $\preccurlyeq_1$, $\preccurlyeq_2$ and $\preccurlyeq_3$.
    E. None of options (A), (B), (C), (D) is correct.

    (Note: This question was incorrectly set. There are 3 answers among the options.)

15. A set S is recursively defined as follows:

    (1) $2 \in S$ and $5 \in S$. (base clause)
    (2) $\forall x, y \in S, x + y \in S$ and $xy \in S$. (recursion clause)
    (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

    Which of the following are true?

    (i) $3 \notin S$ and $4 \notin S$.
    (ii) $|\mathbb{N}| = |S|$.
    (iii) All positive multiples of 5 are elements of S.

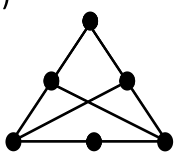
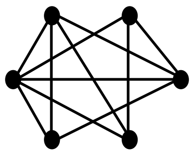
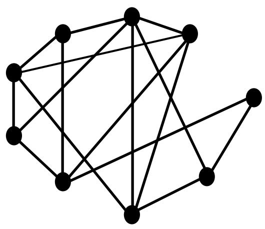
    A. Only (ii).
    B. Only (i) and (ii).
    C. Only (ii) and (iii).
    D. All of (i), (ii) and (iii).
    E. None of options (A), (B), (C), (D) is correct.

16. Which of the following are true?

    (i) The set of all the polynomials with rational coefficients is countable.
    (ii) Two uncountable infinite sets A and B do not have a bijection between them.
    (iii) The set of all strings formed using the alphabet $\Sigma = \{a, b\}$ is uncountable. (Note that strings have finite length.)

    A. Only (i).
    B. Only (ii).
    C. Only (i) and (ii).
    D. Only (ii) and (iii).
    E. None of options (A), (B), (C), (D) is correct.

17. Which of the following graphs are planar?

    (i) 
    (ii) 
    (iii) 

    A. Only (i).
    B. Only (ii).
    C. Only (i) and (ii).
    D. Only (i) and (iii).
    E. None of options (A), (B), (C), (D) is correct.

18. Which of the following statements are true?
    (i) Two spanning trees of a graph will always share some common edges.
    (ii) Every simple planar graph has a vertex of degree at most 5.
    A. Neither (i) nor (ii) is true.
    B. (i) is true but (ii) is false.
    C. (i) is false but (ii) is true.
    D. Both (i) and (ii) are true.

Questions 19 and 20 are based on the information below.

An undirected graph $G = (V, E)$ has n vertices (here $n > 1$), where $V = \{v_1, v_2, \ldots, v_n\}$. The edges are connected based on the following condition:

$\forall i, j \in \{1,2,3, \ldots, n\}$ v_i and v_j are connected $\Leftrightarrow$ $(i + j)$ is a multiple of 5.

Note that $\lfloor x \rfloor$ is the floor function, that is, $\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$.

19. Which of the following are true?
    (i) G has at most 3 connected components.
    (ii) G has a connected component with $\lfloor n/5 \rfloor$ number of vertices if $n \geq 5$.
    (iii) G has 2 connected components with number of vertices in each of those connected components differ by at most 1.
    A. Only (ii).
    B. Only (i) and (iii).
    C. Only (ii) and (iii).
    D. All of (i), (ii) and (iii).
    E. None of options (A), (B), (C), (D) is correct.

20. If n is a multiple of 5, which of the following is correct?
    A. There is a connected component with $2n/5$ number of vertices. No other connected component has the same number of vertices.
    B. There are two connected components with $n/5$ number of vertices each.
    C. There are two connected components with $2n/5$ number of vertices each.
    D. All three connected components have different number of vertices.
    E. None of options (A), (B), (C), (D) is correct.

21. Consider a subset of natural numbers $S = \{x : (2 \nmid x) \wedge (x \leq 22)\}$. What is the smallest natural number n such that in any set of n numbers, chosen from S, there must be two numbers where one divides the other?
    A. 3.
    B. 7.
    C. 8.
    D. 9.
    E. None of options (A), (B), (C), (D) is correct.

22. Three roommates have invited their five other friends for a housewarming party. How many ways can they sit around a circular table so that the three roommates do not sit next to each other? (Note: it is fine for exactly two of the three roommates to sit next to each other, but not okay for all three to sit together.)

    A. $8! - 3!$
    B. $7! - 5! \times 3!$
    C. $8! - 5! \times 3!$
    D. $7! - 4! \times 3!$
    E. None of options (A), (B), (C), (D) is correct.

Part B: There are 4 questions in this part [Total: 56 marks]

23. [Total: 6 marks] Cycle notation.

Denote S_n as a set of all positive integers from 1 up to n inclusive (i.e. $\{1,2,3,\ldots,n\}$). A bijection $\sigma : S_n \to S_n$ is called a *permutation*. A popular way to express permutations is via *cycle notation*.

To write down the permutation $\sigma : S_n \to S_n$ in cycle notation, one proceeds as follows:

1. Write an opening parenthesis followed by an arbitrary element $x \in S_n$. Example: $(\ x$
2. Trace the "orbit" of x, i.e. write down the values under successive applications of σ. Example: $(\ x \quad \sigma(x) \quad \sigma(\sigma(x)) \ \ldots$
3. Repeat until the value returns to x, and close the parenthesis without writing x (again). Example: $\left(x \quad \sigma(x) \quad \sigma(\sigma(x)) \quad \sigma\left(\sigma(\sigma(x))\right)\right)$
4. Continue with an element $y \in S_n$ which was not yet written, and repeat the above process. Example: $\left(x \quad \sigma(x) \quad \sigma(\sigma(x)) \quad \sigma\left(\sigma(\sigma(x))\right)\right)(\ y \ldots)$
5. Repeat until all elements of S_n are written out.

For example, for a permutation $\sigma : S_6 \to S_6$ such that

$$\big(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),\sigma(6)\big) = (2,6,5,4,3,1),$$

its cycle notation would be $(\mathbf{1\ 2\ 6})(\mathbf{3\ 5})(\mathbf{4})$.

(a) Write out the cycle notation for this permutation:
$\sigma : S_4 \to S_4$ such that $\big(\sigma(1),\sigma(2),\sigma(3),\sigma(4)\big) = (3,4,1,2)$. [2 marks]

(b) Write out the cycle notation for this permutation:
$\sigma : S_6 \to S_6$ such that $\big(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),\sigma(6)\big) = (5,4,3,6,1,2)$. [2 marks]

(c) Given two S_6 permutations σ_1 and σ_2 below, what is $\sigma_1 \circ \sigma_2$ in cycle notation?
$\sigma_1 = (1\ 2\ 6)(3)(4)(5)$ and $\sigma_2 = (1)(3\ 6)(2\ 4)(5)$. [2 marks]

24. [Total: 20 marks] Graphs and trees.

(a) An edge weight matrix W of a simple undirected graph has each entry $w_{i,j}$ representing the weight of the edge between vertex i and vertex j. The edge weight matrix of a simple undirected graph is given below. What is the weight of its minimum spanning tree? [2 marks]

$$W = \begin{bmatrix} 0 & 3 & 8 & 3 & 5 \\ 3 & 0 & 4 & 3 & 5 \\ 8 & 4 & 0 & 4 & 6 \\ 3 & 3 & 4 & 0 & 5 \\ 5 & 5 & 6 & 5 & 0 \end{bmatrix}$$

(b) The adjacency matrix A of a directed graph with the vertex set $V = \{v_1, v_2, v_3\}$ is given below, where the rows from top to bottom and columns from left to right are v_1, v_2, v_3 in that order. The entry $a_{i,j}$ represents the number of directed edges from v_i to v_j. Find the number of walks of length 4 from v_1 to v_3. [3 marks]

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

(c) State whether the following statements are true or false. [3 marks]

(i) Graph G has an Euler circuit. If Aiken removes an edge from G, the new graph G' is connected and has an Euler trail.

(ii) There are four non-isomorphic undirected graphs that have two vertices and two edges.

(iii) All simple undirected graphs with n vertices, where $n > 3$, can have degree $n - 2$ for every vertex.

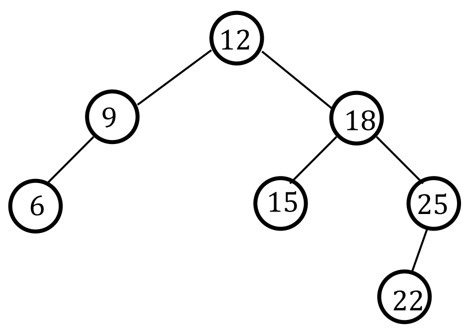
(d) A connected simple undirected graph G has n number of vertices (here n is even). Aiken removes one edge from graph G, and the new graph G' has 2 connected components, both of which are complete graphs. Find the minimum number of edges in G. [4 marks]

(e) The pre-order traversal and post-order traversal of a full binary tree are given below. What is the in-order traversal of the tree? [2 marks]

Pre-order: S U R E C A N D O I T
Post-order: E C R A U O I D T N S

(f) A *Binary Search Tree* (BST) is a specialized form of a binary tree characterized by the following property: for any given vertex with the value x, all elements in its left subtree are smaller than x, and all elements in its right subtree are greater than x. You may assume that there are no duplicate values.

The diagram on the right shows an example of a BST with 7 values.



Find the number of distinct BSTs with (i) 2 values, (ii) 3 values, (iii) 4 values. [3 marks]

(g) State whether the following statements are true or false. [3 marks]

(i) A binary tree has k terminal vertices (leaves) and $k+1$ internal vertices.

(ii) If all the edge weights are distinct in a graph, Prim's and Kruskal's algorithms produce the same minimum spanning tree.

(iii) An edge e is contained in every spanning tree for a connected graph G if and only if removal of e disconnects G.

25. [Total: 20 marks] Counting and probability.

You do not need to show your working for this question.

(a) A loaded coin has the probability of 0.6 for getting a head. It is tossed five times. What is the probability of getting three heads? Write your answer as a single number with four significant digits. [1 mark]

(b) A fair coin is tossed n times, where $n > 10$. Let $1/k$ denote the probability of getting an odd number of heads. What is k? Write your answer as a single number. [2 marks]

(c) Morse code uses dashes and dots to encode the alphanumeric characters. How many distinct messages can be represented using five dashes and three dots? Write your answer as a single number. [2 marks]

(d) What is the total number of ways in which 16 identical tasks can be allotted to 4 distinct processors such that every processor is allotted an even number of tasks? Write your answer as a single number. [3 marks]

(e) An e-commerce platform uses either Apple ID or Google account as a third-party authenticator to register users on the platform. Currently, 85% of the user's log-in use the Google credentials whereas the rest of them use their Apple ID. A welcome bonus of $10 is sent to 95% of the Apple ID registered users. The same bonus is sent to 80% of the Google users. What is the probability that a user has used Google account for authentication given that he or she has received the welcome bonus? Write your final answer as a single number with four significant digits. [3 marks]

(f) You have written a five-digit cheque number on a blank paper and passed the sheet to the cashier in the bank. Note that the number may start with a 0, for example, 00123.
The cashier is unable to tell the number due to the lack of orientation of the blank sheet. How many *distinct* five-digit numbers can be written that can be read both ways: right side up or upside down (i.e. when rotated 180 degrees)? For instance: 09168 can be read as 89160 and vice versa. Write your answer as a single number. [4 marks]

(g) [Total: 5 marks]
How many surjective functions are there from a non-empty finite set A to set B under each of the following conditions?

(i) $|A| = n$ and $|B| = 2$. [2 marks]

(ii) $|A| = n \geq 2$ and $|B| = 3$. [3 marks]

26. [Total: 10 marks] Mathematical induction.

Consider the Fibonacci function:

$$F(0) = 0; \qquad F(1) = 1; \qquad F(n+1) = F(n) + F(n-1), n \geq 1.$$

One interesting property of this function can be expressed as follows:

$$P(a, b) \equiv F(a+b) = \big(F(a+1) \times F(b) + F(a) \times F(b-1)\big),\ \forall a \geq 0, b \geq 1.$$

Use mathematical induction to prove the above property. In your proof, you may use "simplify" for identity/commutative/associative laws of addition and multiplication, but if the distributive law is used, you must state it explicitly.

(a) State all the basis step(s) needed. For example, if you think $P(0,0)$ is the only basis step needed, write $P(0,0)$ as your answer. You do <u>not</u> need to prove your basis step(s). [2 marks]

(b) State and prove all inductive steps. [6 marks]

(c) From this property, state two equations on $F(a+b)$ that can give you a more efficient way of computing Fibonacci numbers. Remember to qualify the conditions of a and b. [2 marks]

=== END OF PAPER ===