NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 2: AY2023/24)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY EIGHT (28)** questions in **TWO (2)** parts and comprises **THIRTEEN (13)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions and write your answers only on the **ANSWER SHEETS** provided.
5. Do **not** write your name on the ANSWER SHEETS.
6. The maximum mark of this assessment is 100.

Question	**Max. mark**
Part A: Q1 – 24	48
Part B: Q25	6
Part B: Q26	6
Part B: Q27	20
Part B: Q28	20
Total	**100**

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions [Total: 24×2 = 48 marks]

Each multiple choice question (MCQ) is worth **TWO marks** and has exactly **one** correct answer.

1. The CS1231S midterm test was conducted on 13 March this semester. Where was it held?

   A. LT15.

   B. Hotel Infinity.

   C. MPSH1.

   D. MPSH2.

   E. This is a trick question; there was no midterm test.

2. The island of Wantuutrewan is inhabited by exactly two types of natives: **knights** who always tell the truth and **knaves** who always lie. Every native is a knight or a knave, but not both.

   You meet four islanders Aiken, Benny, Candy and Dueet. Two of them speak.

   Aiken says: Benny never tells the truth.
   Aiken says: Candy always lies.
   Dueet says: Candy is a knave and I am a knave.

   Which of the following is true?

   A. Both Benny and Candy are knights.

   B. Benny is a knight and Candy is a knave.

   C. Benny is a knave and Candy is a knight.

   D. Both Benny and Candy are knaves.

   E. None of the above, as the puzzle cannot be solved.

3. Define the logical connective $\leadsto$ as follows, where p and q are statement variables:

p	q	$p \leadsto q$
True	True	False
True	False	False
False	True	True
False	False	False

   Which of the following are tautologies, where x, y, z are statement variables?

   (i) $\big((x \leadsto y) \wedge (y \leadsto z)\big) \rightarrow (x \leadsto z)$.

   (ii) $\big((x \leadsto y) \wedge (y \leadsto z)\big) \leadsto (x \leadsto z)$.

   (iii) $\big((x \leadsto y) \wedge (x \leadsto z)\big) \rightarrow x$.

   A. Only (i).

   B. Only (ii).

   C. Only (i) and (ii).

   D. Only (i) and (iii).

   E. None of options (A), (B), (C), (D) is correct.

4. Consider any set A and predicate $P(x, y)$, which of the following statements are true?

   (i) $\forall x \in A\ \forall y \in A\ P(x, y) \rightarrow \exists x \in A\ \exists y \in A\ P(x, y)$.

   (ii) $\exists x \in A\ \exists y \in A\ P(x, y) \rightarrow \forall x \in A\ \forall y \in A\ P(x, y)$.

   (iii) $\forall x \in A\ \exists y \in A\ P(x, y) \rightarrow \exists y \in A\ \forall x \in A\ P(x, y)$.

   (iv) $\exists x \in A\ \forall y \in A\ P(x, y) \rightarrow \forall y \in A\ \exists x \in A\ P(x, y)$.

   A. Only (i).
   B. Only (iv).
   C. Only (i) and (iv).
   D. All of (i), (ii), (iii) and (iv).
   E. None of options (A), (B), (C), (D) is correct.

5. Consider the following statements where A, B are sets and $\mathcal{P}(X)$ denotes the power set of X:

   (i) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

   (ii) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

   (iii) $\mathcal{P}(A \setminus B) = \mathcal{P}(A) \setminus \mathcal{P}(B)$.

   Which of the above statements are true?

   A. Only (i).
   B. Only (iii).
   C. Only (i) and (ii).
   D. Only (i) and (iii).
   E. None of options (A), (B), (C), (D) is correct.

6. Consider the following statements where $A = \{0,1\},\ B = \{-1,1\},\ C = \{-1,0,1\}$:

   (i) $\forall x \in A\ \forall y \in B\ \forall z \in C\ \big((x < y) \rightarrow (y < z)\big)$.

   (ii) $\exists x \in A\ \exists y \in B\ \exists z \in C\ \big((x < y) \wedge (y < z)\big)$.

   (iii) $\forall x \in A\ \exists y \in B\ \big((x^2 \leq y^2) \rightarrow \forall z \in C\ (y \leq z)\big)$.

   Which of the above statements are true?

   A. Only (i).
   B. Only (iii).
   C. Only (i) and (ii).
   D. Only (ii) and (iii).
   E. None of options (A), (B), (C), (D) is correct.

7. Let $f : \mathbb{N} \to \mathbb{N}$ and $f(n) = n(n^2 - 1)$. Which of the following statements are true?

   (i) f is injective.
   (ii) f is surjective.

   A. Only (i).
   B. Only (ii).
   C. Both (i) and (ii).
   D. Neither (i) nor (ii).

8. A set S is recursively defined as follows:

   (1) $1 \in S$ and $2 \in S$. (base clause)
   (2) If $x, y \in S$, then $2x + y \in S$. (recursion clause)
   (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

   What is S?

   A. $\mathbb{N}$.
   B. $\mathbb{Z}^+$.
   C. The set of positive odd integers $\cup \{2\}$.
   D. The set of prime numbers $\cup \{1\}$.
   E. None of the above.

9. A set T is recursively defined as follows:

   (1) $\{a, b\} \in T$ and $\{b, c\} \in T$. (base clause)
   (2) If $X, Y \in T$, then $X \cup Y \in T$ and $X \cap Y \in T$. (recursion clause)
   (3) Membership for T can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

   Which of the following statements is true?

   A. $\emptyset \in T$.
   B. T is a finite set.
   C. T is a countably infinite set.
   D. T is an uncountable set.
   E. None of the above.

10. Aiken wants to use Strong Mathematical Induction to prove the statement below. In the inductive step, he intends to prove $P(k+1)$ by using $P(k-3)$. At least how many base cases does he need to work out?

$$\forall n \in \mathbb{Z}_{\geq 12}\ \exists x, y \in \mathbb{N}\ (n = 4x + 5y).$$

   A. 1
   B. 2
   C. 3
   D. 4
   E. None of the above.

11. Aiken wrote the following proof for the following claim: $\forall n \in \mathbb{N}\ 2n = 0$.

   1. For each $n \in \mathbb{N}$, let $P(n) \equiv 2n = 0$.
   2. Basis step: Clearly, if $n = 0$, then $2n = 0$. Hence $P(0)$ is true.
   3. Inductive step:
      3.1. Let $k \in \mathbb{N}$ such that $P(k)$ is true, i.e., $2k = 0$.
      3.2. Then $k + 1 = i + j$ where i and j are non-negative integers smaller than $k + 1$.
      3.3. Then $2(k+1) = 2(i+j) = 2i + 2j = 0 + 0 = 0$.
      3.4. Hence, $P(k+1)$ is true.
   4. Therefore, $\forall n \in \mathbb{N}\ P(n)$ is true by Strong Mathematical Induction.

   Which of the following statements is correct?

   A. The proof's first flaw is at line 3.1.
   B. The proof's first flaw is at line 3.2.
   C. The proof's first flaw is at line 3.3.
   D. The proof's first flaw is at line 3.4.
   E. The proof is not flawed. Indeed, $\forall n \in \mathbb{N}\ 2n = 0$.

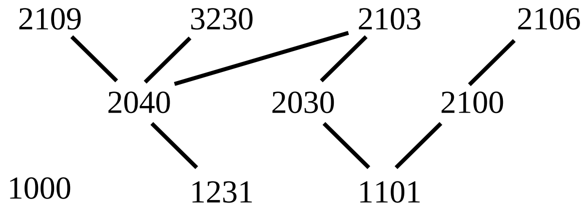
12. Given a relation R on a set $A = \{a, b, c, d\}$ as follows:

$$R = \{(a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (b,d), (c,d)\}.$$

   How many reflexive relations S are there such that $R \subseteq S$?

   A. 2^4
   B. 2^6
   C. 2^8
   D. 2^{10}
   E. None of the above.

13. Consider a partial order $\preccurlyeq$ on $M = \{1000,1101,1231,2030,2040,2100,2103,2106,2109,3230\}$ whose Hasse diagram is shown below:



Which of the following statements are true?

(i) There are exactly 7 elements that are minimal or maximal.

(ii) There is exactly one element that is both minimal and maximal.

(iii) There are exactly 3 elements that are neither minimal nor maximal.

Which of the above statements are true?

A. Only (i).
B. Only (ii).
C. Only (i) and (iii).
D. Only (ii) and (iii).
E. None of options (A), (B), (C), (D) is correct.

14. Let $\preccurlyeq$ be a partial order on a non-empty set A. A subset C of A is called a **chain** if and only if every pair of elements in C are comparable, that is, $\forall a, b \in C\ (a \preccurlyeq b \vee b \preccurlyeq a)$. A **maximal chain** is a chain M such that $t \notin M \Rightarrow M \cup \{t\}$ is not a chain.

Refer to the partial order given in question 13. How many maximal chains are there?

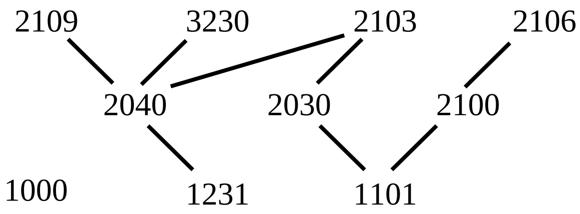
A. 3
B. 4
C. 5
D. 6
E. None of the above.

15. Let $\preccurlyeq$ be a partial order on a non-empty set A. A subset D of A is called an **antichain** if and only if every pair of distinct elements in D are not comparable. A **maximal antichain** is an antichain that is not a proper subset of any other antichain. The **width** of an antichain is one less that the number of elements in it.

Refer to the partial order given in question 13. What is the largest width among all maximal antichains?

A. 3
B. 4
C. 5
D. 6
E. None of the above.

16. Refer to the partial order given in question 13, whose Hasse diagram is reproduced below. How many elements, other than 1231, are comparable to 1231? How many elements, other than 1231, are compatible to 1231? Do <u>not</u> include 1231 in your answers.



A. 1 comparable element and 2 compatible elements.
B. 3 comparable elements and 4 compatible elements.
C. 4 comparable elements and 5 compatible elements.
D. 4 comparable elements and 6 compatible elements.
E. None of the above.

17. Consider a partial order $\preccurlyeq$ on $A = \{1,2,3,4,5,6\}$ defined as follows:

$$\forall x, y \in A \ \left(x \preccurlyeq y \Leftrightarrow \big((x \leq y) \vee (x \mid y)\big)\right).$$

How many possible linearizations of $\preccurlyeq$ are there?

A. 1
B. 6
C. 20
D. 21
E. None of the above.

18. Suppose A and B are finite sets and f and g are functions, which of the following statements are true?

(i) If $|A| > |B|$, then there is no surjection from A to B by the pigeonhole principle.
(ii) If g and $g \circ f$ are surjections, then f must be a surjection.
(iii) If $g \circ f$ is an identity function, then f or g is an identity function, or both f and g are identity functions.

A. Only (i).
B. Only (ii).
C. Only (i) and (ii).
D. Only (ii) and (iii).
E. None of options (A), (B), (C), (D) is correct.

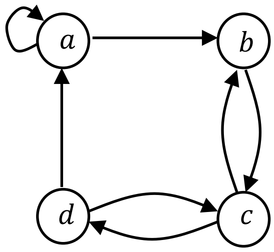
19. Define $f : \mathcal{P}(\mathbb{Z}^+) \setminus \{\emptyset\} \to \mathbb{N}$ by setting $f(S)$ to be the smallest element in S for $S \in \mathcal{P}(\mathbb{Z}^+) \setminus \{\emptyset\}$. Which of the following statements are true?
    (i) The function f has an inverse.
    (ii) $f^{-1}(\{n\})$ is uncountable for some $n \in \mathbb{N}$.
    A. Only (i).
    B. Only (ii).
    C. Both (i) and (ii).
    D. Neither (i) nor (ii).

20. A box contains 5 red balls and 3 blue balls. Dueet picks two balls without replacement from the box. What is the probability that the second ball is red given that the first ball is red?
    A. 1/2
    B. 4/7
    C. 5/8
    D. 2/3
    E. None of the above.

21. Given three events A, B and C, and the following information:
    - A and C are independent
    - B and C are independent
    - A and B are disjoint

    Suppose $P(A \cup C) = \frac{2}{3}$, $P(B \cup C) = \frac{3}{4}$ and $P(A \cup B \cup C) = \frac{11}{12}$. What is $P(C)$?
    A. 1/2
    B. 4/7
    C. 5/8
    D. 2/3
    E. None of the above.

22. How many **walks of length 4** are there from vertex d to vertex b in the directed graph given below?



A. 3
B. 4
C. 5
D. 6
E. None of the above.

23. Which of the following statements are true?

(i) If $G = (V, E)$ is a Hamiltonian graph, then $\forall v \in V \deg(v) \geq 2$.

(ii) Let $G = (V, E)$ be a simple graph such that $\forall v \in V \deg(v) \geq \left\lfloor \frac{|V|}{2} \right\rfloor$, then G is connected. (Note: $\lfloor x \rfloor$ is the floor function.)

(iii) Any complete bipartite graph $K_{2,n}$ where $n > 5$, is planar.

A. Only (i) and (ii).
B. Only (i) and (iii).
C. Only (ii) and (iii).
D. All of (i), (ii) and (iii).
E. None of options (A), (B), (C), (D) is correct.

24. Dueet drew 20 labelled trees with 5 vertices. There will be at least how many trees that are isomorphic to one another? You are to choose the tightest bound.

A. 4
B. 6
C. 7
D. 9
E. None of the above.

Part B: There are 4 questions in this part [Total: 52 marks]

25. [Total: 6 marks]

Let R be a non-empty relation on $A = \{a, b\}$, and let S be any relation on A such that $R \subseteq S$. For each of the following statements, state whether it is TRUE or FALSE (do not write T or F). If you answer TRUE, you do not need to provide a proof. If you answer FALSE, you must provide a counterexample, clearly indicating what your R and S are. If your counterexample is missing or incorrect, no mark will be awarded. Do not draw diagrams, as they will be disregarded.

(a) If R is reflexive, then S is reflexive.
(b) If R is irreflexive, then S is irreflexive.
(c) If R is symmetric, then S is symmetric.
(d) If R is transitive, then S is transitive.
(e) If R is antisymmetric, then S is antisymmetric.
(f) If R is asymmetric, then S is asymmetric.

26. [Total: 6 marks]

Recall Theorem 4.4.1 (The Quotient-Remainder Theorem):

> "Given any integer n and positive integer d, there exist unique integers q and r such that $n = dq + r$ and $0 \le r < d$."

Definitions:

Let $n \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$. Suppose $n = dq + r$ and $0 \le r < d$, we define $n \mathbin{\%} d = r$.

Given a function $f(x)$, we define the function $\boldsymbol{f^{(n)}(x)}$ to be the result of n applications of f to x, where $n \in \mathbb{Z}^+$. For example, $f^{(3)}(x) = f\left(f\big(f(x)\big)\right)$. We also define the ***order*** of an input x with respect to f to be the smallest positive integer m such that $f^{(m)}(x) = x$.

Define a function $g : A \to A$ by setting, for each $x \in A$, $g(x) = 3x \mathbin{\%} 5$.

You do not need to show your working for the parts below.

(a) What is $g^{(3)}(21)$? [1 mark]

(b) What is the order of 3 with respect to the function g? [2 marks]

(c) Let $A = \{0,1,2,3,4\}$. Define the relation R on A as follows:

$x \; R \; y$ iff the order of x is equal to the order of y with respect to the function g.

R is an equivalence relation. Write out all the distinct equivalence classes of R using set-roster notation. Do not use the [ ] notation. [3 marks]

27. [Total: 20 marks]

Workings are not required for this question except for part (g).

(a) A bag contains 9 white balls and 5 black balls. What is the probability of picking two balls of different colours without replacement? Write your answer as a single fraction. [3 marks]

(b) A standard deck of playing cards consists of 52 cards, with 13 cards of each of the four suits. If you draw 6 cards from the deck without replacement, what is the probability of drawing no more than 2 spades? Write your answer in percentage with 3 significant figures. [3 marks]

(c) Given a loaded 6-sided die, the probability of rolling each number is in a direct ratio with its value, that is, the probability of rolling the number n is n/m times the probability of rolling the number m. What is the probability of rolling an even number? Write your answer as a single fraction. [3 marks]

(d) A bulb factory has 3 machines *A*, *B* and *C*, manufacturing respectively 25%, 35% and 40% of the bulbs. Machines *A*, *B* and *C* have respectively 5%, 4% and 2% of defective output. A bulb is chosen at random and found to be defective. What is the probability that it was manufactured by machine *A*? Write your answer as a single fraction. [3 marks]

(e) Two fair 6-sided dice are tossed. The absolute difference is the difference in the numbers shown on the dice in absolute value. For example, if one die shows a 2 and the other a 5, then the absolute difference is 3. What is the expected value of the absolute difference of two dice? Write your answer as a single fraction. [3 marks]

(f) How many solutions are there for the equation below, where x, y, z are non-negative integers? Write your answer as a single integer.

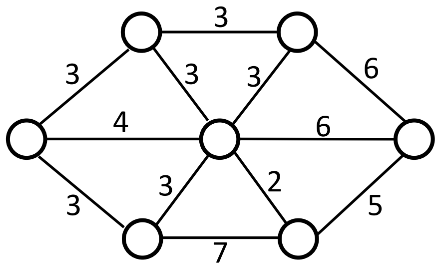
$$x + y + z = 6$$

[1 mark]

(g) How many ways can you write 1000000 (one million) as a product of 3 non-negative integers, if the order of integers in the product matters? Write your answer as a single integer <u>and show your working or explain your answer</u>. (Hint: You may make use of part (f) above.) [4 marks]

28. [Total: 20 marks]

(a) An undirected weighted graph is shown below. What is the weight of its minimum spanning tree? [2 marks]



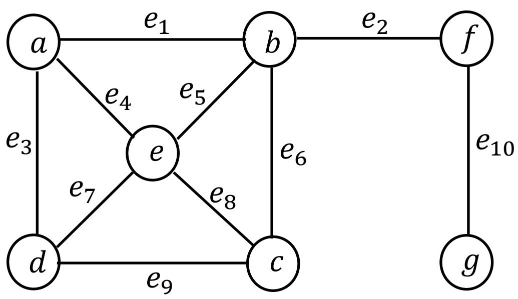
(b) Referring to the undirected weighted graph in part (a) above, how many non-isomorphic minimum spanning trees are there? [3 marks]

(c) The preorder and inorder traversals of a certain binary tree are given below. Write out the postorder traversal of this tree. [3 marks]

Preorder: S U R E C A N D O I T
Inorder: R U A C E S D N I O T

(d) Let G be a connected graph. A vertex is a **cut-vertex** such that removing it and all edges incident to it disconnects G, that is, increasing the number of components in G.

List out any two cut-vertices in the graph below. [2 marks]



(e) Let G be a connected graph. A **cutset** S of G is a minimal set of edges of G such that the removal of all the edges in S from G disconnects G, and no proper subset of S is a cutset.

Hence, for the graph in part (d), $\{e_3, e_7, e_9\}$ is a cutset but $\{e_3, e_4, e_8, e_7, e_9\}$ is not, as $\{e_3, e_7, e_9\}$ is a proper subset of $\{e_3, e_4, e_8, e_7, e_9\}$.

List out any three cutsets in the graph in part (d) above, such that each cutset disconnects G into components with at least 2 vertices in each component. Your three cutsets must have different cardinalities. [3 marks]

(f) Let $G = (V, E)$ be an undirected graph, and let $U \subseteq V$. The induced subgraph $G[U]$ is a graph whose vertex set is U and whose edge set consists of all the edges in E that have both endpoints in U. That is, for any two vertices $x, y \in U$, x and y are adjacent in $G[U]$ if and only if they are adjacent in G.

Figure 0 shows a graph G and Figures 1-5 show five subgraphs of G, namely, G_1, G_2, G_3, G_4 and G_5. Which of G_1, G_2, G_3, G_4 and G_5 are induced subgraphs of G? [3 marks]

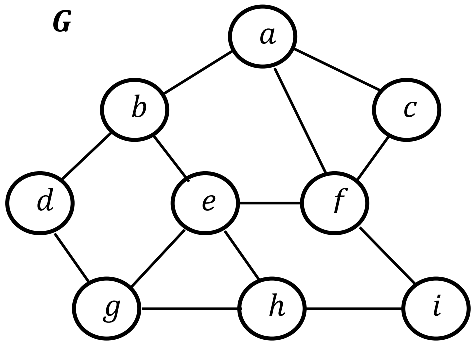


Figure 0

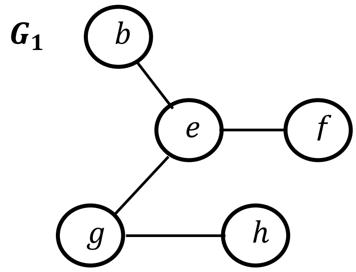


Figure 1

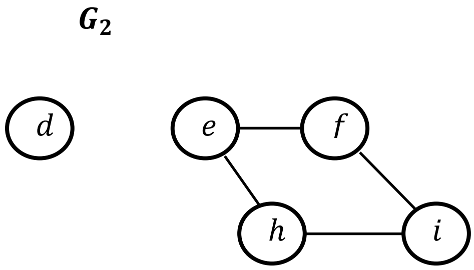


Figure 2

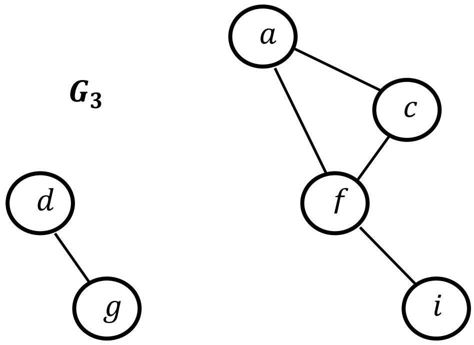


Figure 3

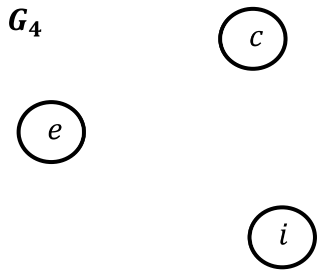


Figure 4

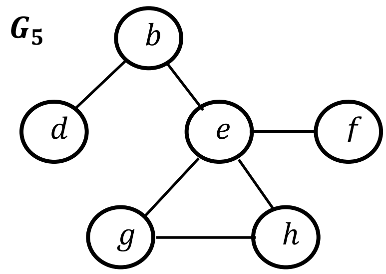


Figure 5

(g) In a country with 30 cities, every city is connected to every other city with a two-way road. At most how many roads can be closed such that you can still get from any city to another city? Explain how you get your answer. [4 marks]

=== END OF PAPER ===