3. The Logic of Quantified Statements (aka Predicate Logic)

Aaron Tan

AY2024/25 Semester 1

3.1 Predicates and Quantified Statements I

- Predicate; domain; truth set
- Universal quantifier $\forall$, existential quantifiers $\exists$ and $\exists!$
- Universal conditional statements; Implicit quantification

3.2 Predicates and Quantified Statements II

- Negation of quantified statements; negation of universal conditional statements
- Vacuous truth of universal statements
- Variants of universal conditional statements (contrapositive, converse, inverse)
- Necessary and sufficient conditions, only if

3.3 Statements with Multiple Quantifiers

- Negations of multiply-quantified statements; order of quantifiers
- Prolog

3.4 Arguments with Quantified Statements

- Universal instantiation; universal modus ponens; universal modus tollens

Reference: Epp's Chapter 3 The Logic of Quantified Statements

3.1 Predicates and Quantified Statements I

Proposition and Predicate

In the previous lecture, we introduced statements (or propositions). A statement (or proposition) can be true or false, but not both.

Examples:

- $2 + 3 = 5$
- $\sqrt{5} < 2$

What about these?

- $x + 3 = 5$
- $\sqrt{n} < 2$

A statement's (or proposition's) truth value does not depend on any variables. A predicate's truth value depends on the variable(s) in it.

3.1.1. Predicates and Quantified Statements I

In logic, predicates can be obtained by removing some or all of the nouns from a statement. For instance, let P stand for "is a student at NUS" and let Q stand for "is a student at." Then both P and Q are *predicate symbols*.

Predicate variables:

$P(x)$ = "x is a student at NUS"
$Q(x, y)$ = "x is a student at y"

When concrete values are substituted in place of predicate variables, a statement results.

For simplicity, we define a predicate to be a predicate symbol together with suitable predicate variables.

In some treatments of logic, such objects are referred to as propositional functions or open sentences.

Definition 3.1.1 (Predicate)

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

"Domain" may also be known as "domain of discourse", "universe of discourse", "universal set", or simply "universe". The last three terms are usually used in set theory.

When an element in the domain of the variable of a one-variable predicate is substituted for the variable, the resulting statement is either true or false. The set of all such elements that make the predicate true is called the truth set of the predicate.

Definition 3.1.2 (Truth set)

If $P(x)$ is a predicate and x has domain D, the **truth set** is the set of all elements of D that make $P(x)$ true when they are substituted for x.
The truth set of $P(x)$ is denoted $\{x \in D \mid P(x)\}$.

In set theory, the symbol $\mid$ is used to mean “such that”.

Let $Q(n)$ be the predicate "n is a factor of 8."

Find the truth set of $Q(n)$ if

a. the domain of n is the set $\mathbb{Z}^+$.

{1, 2, 4, 8} because these are exactly the positive integers that divide 8 evenly.

b. the domain of n is the set $\mathbb{Z}$.

{1, 2, 4, 8, -1, -2, -4, -8}

3.1.2. The Universal Quantifier: ∀

One sure way to change predicates into statements is to assign specific values to all their variables.

Example: If x represents the number 35, the sentence "x is divisible by 5" is a true statement.

Another way to obtain statements from predicates is to add quantifiers.

Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.

The symbol ∀ denotes "for all" (or "for any", "for every", "for each") and is called the universal quantifier.

Definition 3.1.3 (Universal Statement)

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$".

- It is defined to be true iff $Q(x)$ is true for every x in D.
- It is defined to be false iff $Q(x)$ is false for at least one x in D.

A value for x for which $Q(x)$ is false is called a **counterexample.**

Truth and Falsity of Universal Statements

a. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D\ (x^2 \geq x).$$

Show that this statement is **true**.

Check that "$x^2 \geq x$" is true for each x in D.

$1^2 \geq 1$, $2^2 \geq 2$, $3^2 \geq 3$, $4^2 \geq 4$, $5^2 \geq 5$.

Hence "$\forall x \in D\ (x^2 \geq x)$" is true.

This method is called the method of exhaustion.

Truth and Falsity of Universal Statements

b. Consider the statement

$$\forall x \in \mathbb{R}\ (x^2 \geq x).$$

Find a counterexample to show that this statement is **false**.

Take $x = ½$. Then x is in $\mathbb{R}$ and

$$(½)^2 = ¼ \ngeq ½ .$$

Hence "$\forall x \in \mathbb{R}\ (x^2 \geq x)$" is false.

3.1.3. The Existential Quantifier: ∃

Example: “There is a student in CS1231S” can be written as

∃ a person *p* such that *p* is a student in CS1231S.

Or, more formally,

$\exists p \in P$ such that *p* is a student in CS1231S.

where *P* is the set of all people.

- The words *such that* are inserted just before the predicate. If the context is clear, sometimes the abbreviation s.t. is used.
- Some alternative expressions for “there exists” are “there is a”, “we can find a”, “there is at least one”, “for some”, and “for at least one”.

Sentences that are quantified existentially are defined as statements by giving them the truth values specified in the following definition.

Definition 3.1.4 (Existential Statement)

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$".

- It is defined to be true iff $Q(x)$ is true for at least one x in D.
- It is defined to be false iff $Q(x)$ is false for all x in D.

Truth and Falsity of Existential Statements

a. Show that the following statement is true.

$$\exists m \in \mathbb{Z}^+ \text{ such that } m^2 = m.$$

Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence "$\exists m \in \mathbb{Z}^+$ such that $m^2 = m$" is true.

b. Let $E = \{5, 6, 7, 8\}$. Show that the following statement is false.

$$\exists m \in E \text{ such that } m^2 = m.$$

Note that $m^2 = m$ is not true for any integer m from 5 through 8: $5^2 = 25 \neq 5$, $6^2 = 36 \neq 6$, $7^2 = 49 \neq 7$, $8^2 = 64 \neq 8$.

Hence "$\exists m \in E$ such that $m^2 = m$" is false.

3.1.3. The Existential Quantifier: ∃!

The symbol ∃! is used to denote “there exists a unique” or “there is one and only one”.

Example: $\exists!\, x \in \mathbb{Z}^{+}$ such that x is even and prime.

3.1.4. Formal Versus Informal Language

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol ∀ or ∃.

a. $\forall x \in \mathbb{R},\ x^2 \geq 0.$

- All real numbers have non-negative squares.
- Every/Any real number has a non-negative square.

b. $\forall x \in \mathbb{R},\ x^2 \neq -1.$

- All real numbers have squares that are not -1.
- No real numbers have squares equal to -1.

c. $\exists m \in \mathbb{Z}^+$ such that $m^2 = m$.

- There is a positive integer whose square is itself.
- Some positive integer equals its own square.

With experience, you may omit commas and "such that".
Eg: "$\forall x \in \mathbb{R}\ x^2 \geq 0$", "$\forall x \in \mathbb{R}\ (x^2 \geq 0)$", "$\exists m \in \mathbb{Z}^+\ m^2 = m$".

3.1.5. Universal Conditional Statements

A reasonable argument can be made that the most important form of statement in mathematics is the universal conditional statement:

$$\forall x \text{ (if } P(x) \text{ then } Q(x)).$$

or

$$\forall x \, (P(x) \rightarrow Q(x)).$$

Familiarity with statements of this form is essential if you are to learn to speak mathematics.

3.1.6. Equivalent Forms of Universal and Existential Statements

Are these two statements the same?

$\forall$ real numbers *x*, if *x* is an integer then *x* is rational.

$\forall$ integers *x*, *x* is rational.

Yes, they have the same informal translation:

All integers are rational.

By narrowing *U* to be the domain *D* consisting of all values of the variable *x* that make *P*(*x*) true,

$\forall x \in U\ (P(x) \rightarrow Q(x))$ ⇒ $\forall x \in D,\ Q(x)$

Rewrite the statement "All squares are rectangles" in the two forms:

- $\forall x$, (if *x* is a square then *x* is a rectangle).
- $\forall$ squares *x*, *x* is a rectangle .

Similarly,

$\exists x \in U$ such that $(P(x)$ and $Q(x))$

↓

$\exists x \in D$ such that $Q(x)$

where D is the set of all x for which $P(x)$ is true.

A **prime number** is an integer whose only positive integer factors are itself and 1. Consider the statement

"There is an integer that is both prime and even."

Let Prime(n) be "n is prime" and Even(n) be "n is even". Use the notation Prime(n) and Even(n) to rewrite this statement into the following two forms:

a. $\exists n$ such that (Prime(n) $\wedge$ Even(n)).

b. $\exists$ an even number n such that Prime(n) .

3.1.7. Implicit Quantification

Mathematical writing contains many examples of implicitly quantified statements. Some occur, through the presence of the word *a* or *an*. Others occur in cases where the general context of a sentence supplies part of its meaning.

For example, in an algebra course in which the letter x is always used to indicate a real number, the predicate

$$\text{If } x > 2 \text{ then } x^2 > 4$$

is interpreted to mean the same as the statement

$$\forall \text{ real numbers } x, (\text{if } x > 2 \text{ then } x^2 > 4).$$

3.1.8. Tarski's World

Tarski's World is a computer program developed by information scientists Jon Barwise and John Etchemendy to help teach the principles of logic.

It is described in their book *The Language of First-Order Logic*, which is accompanied by a CD-ROM containing the program Tarski's World, named after the great logician Alfred Tarski.

The program for Tarski's World provides pictures of blocks of various sizes, shapes, and colors, which are located on a grid.

- Shown in Figure 3.1.1 is a picture of an arrangement of objects in a two-dimensional Tarski world.

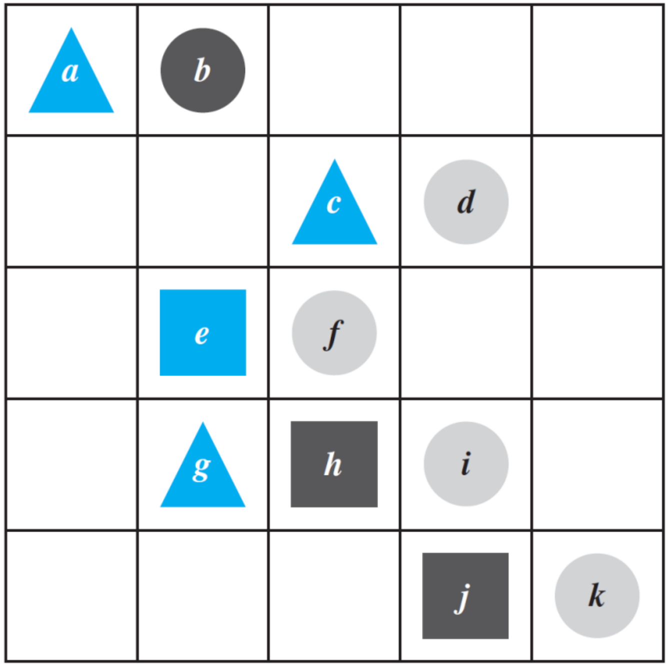


Figure 3.1.1

The configuration can be described using logical operators and — for the two-dimensional version — notation such as:

- Triangle(*x*), meaning "*x* is a triangle,"
- Blue(*y*), meaning "*y* is blue," and
- RightOf(*x*, *y*), meaning "*x* is to the right of *y* (but possibly in a different row)."

Individual objects can be given names such as *a*, *b*, or *c*.

Determine the truth or falsity of the following statements. The domain for all variables is the set of objects in the Tarski's world shown on the right.

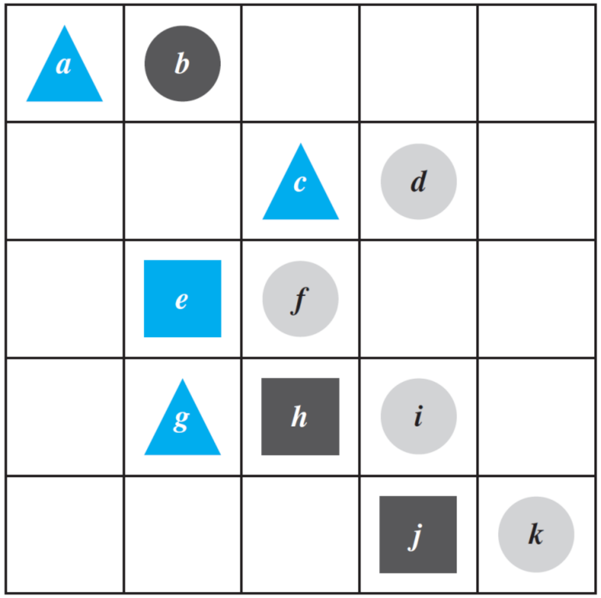


Figure 3.1.1

a. $\forall t\,(\text{Triangle}(t) \rightarrow \text{Blue}(t))$. True

b. $\forall x\,(\text{Blue}(x) \rightarrow \text{Triangle}(x))$. False

c. $\exists y$ such that $(\text{Square}(y) \wedge \text{RightOf}(d, y))$. True

d. $\exists z$ such that $(\text{Square}(z) \wedge \text{Gray}(z))$. False

3.2 Predicates and Quantified Statements II

3.2.1. Negations of Quantified Statements

Theorem 3.2.1 Negation of a Universal Statement

The **negation** of a statement of the form

$$\forall x \in D, P(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D \text{ such that } \sim P(x)$$

Symbolically,

$$\sim(\forall x \in D, P(x)) \equiv \exists x \in D \text{ such that } \sim P(x)$$

That is, the negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Theorem 3.2.2 Negation of an Existential Statement

The **negation** of a statement of the form

$$\exists x \in D \text{ such that } P(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D, \sim P(x)$$

Symbolically,

$$\sim(\exists x \in D \text{ such that } P(x)) \equiv \forall x \in D, \sim P(x)$$

That is, the negation of an existential statement (“some are”) is logically equivalent to a universal statement (“none are” or “all are not”).

Write formal negations for the following statements:

a. ∀ primes *p*, *p* is odd.

∃a prime *p* such that *p* is not odd.

b. ∃ a triangle *T* such that the sum of the angles of *T* equals 200°.

∀ triangles *T*, the sum of the angles of *T* does not equal 200°.

3.2.2. Negations of Universal Conditional Statements

Of special importance in mathematics.

$\sim(\forall x\ (P(x) \rightarrow Q(x))) \equiv \exists x$ such that $\sim(P(x) \rightarrow Q(x))$ … (A)

$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x)$ … (B)

Substituting (B) into (A):

$\sim(\forall x\ (P(x) \rightarrow Q(x))) \equiv \exists x$ such that $(P(x) \wedge \sim Q(x))$

Negations of Universal Conditional Statements: Quick Quiz

Write a formal negation for statement (a) and an informal negation for statement (b):

a. ∀ people *p*, if *p* is blond then *p* has blue eyes.

∃a person *p* such that *p* is blond and *p* does not have blue eyes.

b. If a computer program has more than 100,000 lines, then it contains a bug.

There is at least one computer program that has more than 100,000 lines and does not contain a bug.

3.2.3. The Relation among ∀, ∃, ∧, and ∨

- Analogous to De Morgan's laws, which state that the negation of an *and* statement is an *or* statement and that the negation of an *or* statement is an *and* statement.
- This similarity is not accidental. In a sense, universal statements are generalizations of *and* statements, and existential statements are generalizations of *or* statements.

The Relation among $\forall$, $\exists$, $\wedge$, and $\vee$

If $Q(x)$ is a predicate and the domain D of x is the set $\{x_1, x_2, \ldots, x_n\}$, then

$$\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$$

Similarly,

$$\exists x \in D, Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$$

3.2.4. Vacuous Truth of Universal Statements

- Suppose a bowl sits on a table and next to the bowl is a pile of five blue and five gray balls, any of which may be placed in the bowl.
- If three blue balls and one gray ball are placed in the bowl, as shown in Figure 3.2.1(a), the statement "All the balls in the bowl are blue" would be false (since one of the balls in the bowl is gray).

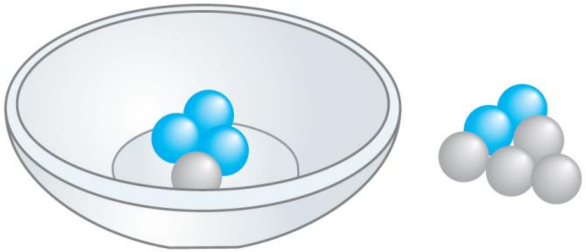


Figure 3.2.1(a)

- Now suppose that no balls at all are placed in the bowl, as shown in Figure 3.2.1(b).
- Consider the statement:

All the balls in the bowl are blue.

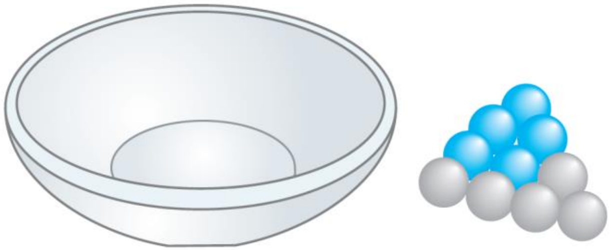


Figure 3.2.1(b)

- Now, is the statement "All the balls in the bowl are blue" true or false?

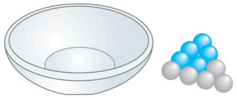


Figure 3.2.1(b)

- The statement is false if, and only if, its negation is true.
- And its negation is: There exists a ball in the bowl that is not blue.
- But the only way this negation can be true is for there actually to be a non-blue ball in the bowl.
- And there is not! Hence the negation is false, and so the statement is true "by default".

In general, a statement of the form

$$\forall x \in D\ (P(x) \rightarrow Q(x))$$

is called vacuously true or true by default if, and only if, $P(x)$ is false for every x in D.

- A vacuous truth is a conditional or universal statement that is only true because the hypothesis (antecedent) cannot be satisfied.
- For this reason, sometimes we say a statement is vacuously true only because it does not really say anything.

$\forall a \in X, P(a)$ is vacuously true if X is an empty set.
(Eg: All mooloomeelees are mammals.)

Definition: A set A is a subset of set B, denoted as $A \subseteq B$, if every element in A is an element in B.

Proof that the empty set $\emptyset$ is a subset of every set.

Proof: Since $\forall x, (x \notin \emptyset)$, the argument holds vacuously.

(Alternatively can prove by contradiction, but is longer.)

3.2.5. Variants of Universal Conditional Statements

We have known that a conditional statement has a contrapositive, a converse, and an inverse.

The definitions of these terms can be extended to universal conditional statements.

Definition 3.2.1 (Contrapositive, converse, inverse)

Consider a statement of the form: $\forall x \in D\ (P(x) \rightarrow Q(x))$.

1. Its **contrapositive** is: $\forall x \in D\ ({\sim}Q(x) \rightarrow {\sim}P(x))$.
2. Its **converse** is: $\forall x \in D\ (Q(x) \rightarrow P(x))$.
3. Its **inverse** is: $\forall x \in D\ ({\sim}P(x) \rightarrow {\sim}Q(x))$.

Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

The formal version: $\forall x \in \mathbb{R}\ (x > 2 \rightarrow x^2 > 4)$.

Contrapositive: $\forall x \in \mathbb{R}\ (x^2 \leq 4 \rightarrow x \leq 2)$.

If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

Converse: $\forall x \in \mathbb{R}\ (x^2 > 4 \rightarrow x > 2)$.

If the square of a real number is greater than 4, then the number is greater than 2.

Inverse: $\forall x \in \mathbb{R}\ (x \leq 2 \rightarrow x^2 \leq 4)$.

If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.

Let $P(x)$ and $Q(x)$ be any predicates, let D be the domain of x, and consider the statement:

$$\forall x \in D\ (P(x) \rightarrow Q(x))$$

and its contrapositive

$$\forall x \in D\ (\sim Q(x) \rightarrow \sim P(x))$$

Any particular x in D that makes "$P(x) \rightarrow Q(x)$" true also makes "$\sim Q(x) \rightarrow \sim P(x)$" true (by the logical equivalence between $p \rightarrow q$ and $\sim q \rightarrow \sim p$).

It follows that "$P(x) \rightarrow Q(x)$" is true for all x in D iff "$\sim Q(x) \rightarrow \sim P(x)$" is true for all x in D.

$$\forall x \in D\ (P(x) \rightarrow Q(x)) \equiv \forall x \in D\ (\sim Q(x) \rightarrow \sim P(x))$$

Consider the statement:

$$\forall x \in \mathbb{R}\ (x > 2 \rightarrow x^2 > 4)$$ True

and its converse

$$\forall x \in \mathbb{R}\ (x^2 > 4 \rightarrow x > 2)$$ False

A universal conditional statement is not logically equivalent to its converse.

$$\forall x \in D\ (P(x) \rightarrow Q(x)) \not\equiv \forall x \in D\ (Q(x) \rightarrow P(x))$$

3.2.6. Necessary and Sufficient Conditions, Only if

The definitions of necessary, sufficient, and only if can also be extended to apply to universal conditional statements.

Definition 3.2.2 (Necessary and Sufficient conditions, Only if)

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x\,(r(x) \rightarrow s(x))$".
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x\,(\sim r(x) \rightarrow \sim s(x))$" or, equivalently, "$\forall x\,(s(x) \rightarrow r(x))$".
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x\,(\sim s(x) \rightarrow \sim r(x))$" or, equivalently, "$\forall x\,(r(x) \rightarrow s(x))$" .

Rewrite the following statements as quantified conditional statements. Do not use the word *necessary* or *sufficient*:

a. Squareness is a sufficient condition for rectangularity.

$\forall x$, if x is a square, then x is a rectangle.

Informal: If a figure is a square, then it is a rectangle.

b. Being at least 35 years old is a necessary condition for being President of the United States.

$\forall$people x, if x is younger than 35, then x cannot be President of the United States.

or

$\forall$people x, if x is President of the United States, then x is at least 35 years old.

Common beginners' mistakes

Given the following predicates:

- Bird(x): x is a bird
- Fly(x): x can fly

1. Write a quantified statement for the following sentence:

   All birds can fly.

Answer:

$\forall x, \text{Fly}(\text{Bird}(x))$ ✗

Why? Bird(x) is a **predicate**; it evaluates to true or false. This is like writing Fly(true) or Fly(false)!

$\forall x, (\text{Bird}(x) \wedge \text{Fly}(x))$ ✗

Why? This is saying everything must be a bird and it flies!

$\forall x, (\text{Bird}(x) \rightarrow \text{Fly}(x))$ ✓

Common beginners' mistakes

2. Write a quantified statement for the following sentence:

   There is a bird that can fly.

Answer:

$\exists x$ s.t. $(\text{Bird}(x) \rightarrow \text{Fly}(x))$ ✗

What if there are no birds at all?

$\exists x$ s.t. $(\text{Bird}(x) \wedge \text{Fly}(x))$ ✓

Common beginners' mistakes

3. Write a quantified statement for the following sentence (do not begin with negated quantifier, such as $\sim\forall$ or $\sim\exists$):

   Not all birds can fly.

Answer:

$\forall x, (\mathrm{Bird}(x) \rightarrow {\sim}\mathrm{Fly}(x))$ ✗ — This means all birds can't fly!

$\exists x \text{ s.t. } (\mathrm{Bird}(x) \rightarrow {\sim}\mathrm{Fly}(x))$ ✗ — Again, what if there are no birds?

$\exists x \text{ s.t. } (\mathrm{Bird}(x) \wedge {\sim}\mathrm{Fly}(x))$ ✓

Check: From Q1, "all birds can fly" $\equiv$ "$\forall x, (\mathrm{Bird}(x) \rightarrow \mathrm{Fly}(x))$".
$\therefore$ "Not all birds can fly" $\equiv$ "$\sim(\forall x, (\mathrm{Bird}(x) \rightarrow \mathrm{Fly}(x)))$" $\equiv$ "$\sim(\forall x, ({\sim}\mathrm{Bird}(x) \vee \mathrm{Fly}(x)))$"
$\equiv$ "$\exists x \text{ s.t. } {\sim}({\sim}\mathrm{Bird}(x) \vee \mathrm{Fly}(x))$" $\equiv$ "$\exists x \text{ s.t. } (\mathrm{Bird}(x) \wedge {\sim}\mathrm{Fly}(x))$".

3.3 Statements with Multiple Quantifiers

Statements with Multiple Quantifiers

Consider the Tarski's world again.

Show that the following statement is true:

For all triangles *x*, there is a square *y* such that *x* and *y* have the same color.

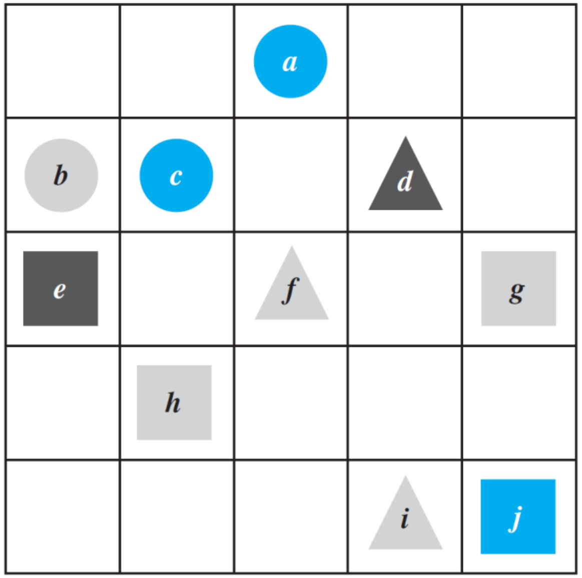


Figure 3.3.1

The statement says that no matter which triangle someone gives you, you will be able to find a square of the same color.

There are only 3 triangles *d*, *f*, and *i*.

Given $x =$	choose $y =$	and check that y is the same color as x.
d	e	yes •
f or i	h or g	yes •

3.3.1. Interpreting Multiply-Quantified Statements

If you want to establish the truth of a statement of the form:

$$\forall x \in D,\ \exists y \in E \text{ such that } P(x, y)$$

your challenge is to allow someone else to pick whatever element x in D they wish and then you must find an element y in E that "works" for that particular x.

If you want to establish the truth of a statement of the form:

$$\exists x \in D \text{ such that } \forall y \in E,\ P(x, y)$$

your job is to find one particular x in D that will "work" no matter what y in E anyone might choose to challenge you with.

A college cafeteria line has four stations: salads, main courses, desserts, and beverages.

The salad station offers a choice of green salad or fruit salad; the main course station offers spaghetti or fish; the dessert station offers pie or cake; and the beverage station offers milk, soda, or coffee. Three students, Uta, Tim, and Yuen, go through the line and make the following choices:

- Uta: green salad, spaghetti, pie, milk
- Tim: fruit salad, fish, pie, cake, milk, coffee
- Yuen: spaghetti, fish, pie, soda

These choices are illustrated in Figure 3.3.2.

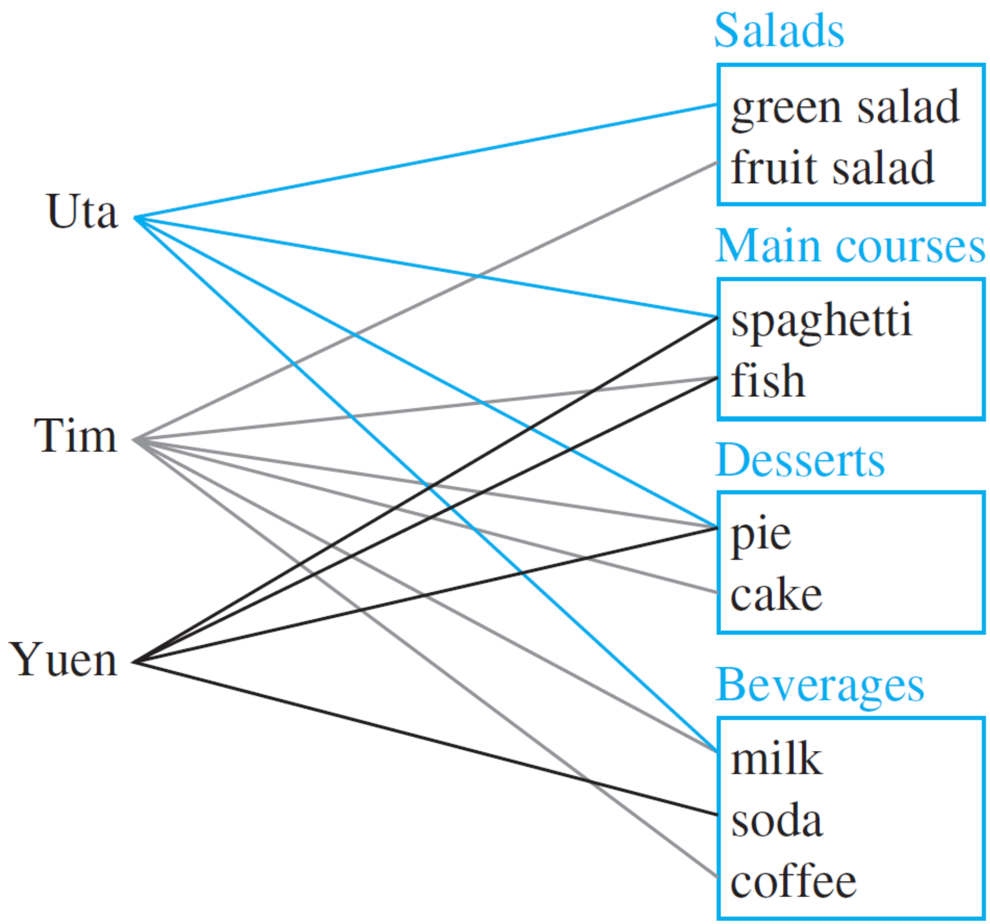


Figure 3.3.2

Interpreting Multiply-Quantified Statements

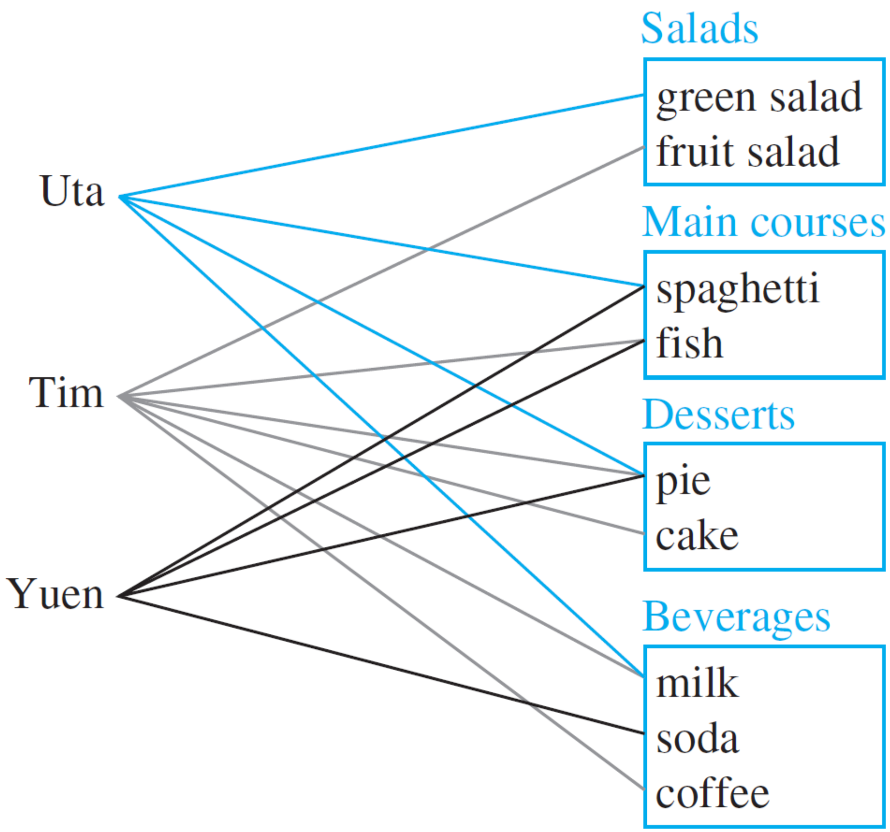


Figure 3.3.2

For all students and stations, there is an item such that every student chose that item.

False.

Write each of following statements informally and find its truth value.

a. ∃ an item *I* such that ∀ students *S*, *S* chose *I*.

There is an item that was chosen by every student. True (pie).

b. ∃ a student *S* such that ∀ items *I*, *S* chose *I*.

There is a student who chose every available item. False.

c. ∃ a student *S* such that ∀ stations *Z*, ∃ an item *I* in *Z* such that *S* chose *I*.

There is a student who chose at least one item from every station. True (Uta and Tim).

d. ∀ students *S* and ∀ stations *Z*, ∃ an item *I* in *Z* such that *S* chose *I*.

3.3.2. Translating from Informal to Formal Language

Most problems are stated in informal language, but solving them often requires translating them into more formal terms.

Example: The **reciprocal** of a real number *a* is a real number *b* such that *ab* = 1. The following 2 statements are true. Rewrite them formally using quantifiers and variables:

a. Every nonzero real number has a reciprocal.

$\forall$ nonzero real numbers *u*, $\exists$ a real number *v* such that *uv* = 1.

b. There is a real number with no reciprocal.

$\exists$ a real numbers *c* such that $\forall$ real number *d*, $cd \neq 1$.

3.3.3. Ambiguous Language

You are visiting a computer microchips factory. The factory guide tells you:

There is a person supervising every detail of the production process.

"there is" – existential quantifier; "every" – universal quantifier.

Which of the following best describes its meaning?

- There is one single person who supervises all the details of the production process.
- For any particular production detail, there is a person who supervises the detail, but there might be different supervisors for different details.

Once you interpreted the sentence in one way, it may have been hard for you to see that it could be understood in the other way.

Perhaps you had difficulty even though the two possible meanings were explained.

Although statements written informally may be open to multiple interpretations, we cannot determine their truth or falsity without interpreting them one way or another.

Therefore, we have to use context to try to ascertain their meaning as best we can.

3.3.4. Negations of Multiply-Quantified Statements

Recall in 3.2.1:

$\sim(\forall x \in D, P(x)) \equiv \exists x \in D$ such that $\sim P(x)$

$\sim(\exists x \in D$ such that $P(x)) \equiv \forall x \in D, \sim P(x)$

(A) So, to find: $\sim(\forall x \in D, \exists y \in E$ such that $P(x, y))$

➔ $\exists x \in D$ such that $\sim(\exists y \in E$ such that $P(x, y))$

➔ $\exists x \in D$ such that $\forall y \in E, \sim P(x, y)$.

$\sim(\forall x \in D, \exists y \in E$ such that $P(x, y))$ ≡ $\exists x \in D$ such that $\forall y \in E, \sim P(x, y)$

(B) Similarly, to find: $\sim(\exists x \in D$ such that $\forall y \in E, P(x, y))$

➔ $\forall x \in D, \sim(\forall y \in E, P(x, y))$

➔ $\forall x \in D, \exists y \in E$ such that $\sim P(x, y)$.

$\sim(\exists x \in D$ such that $\forall y \in E, P(x, y))$ ≡ $\forall x \in D, \exists y \in E$ such that $\sim P(x, y)$

Refer to the Tarski's world of Figure 3.3.1 again.

Write a negation for each of the following statements, and determine which is true, the given statement or its negation.

a. For all squares *x*, there is a circle *y* such that *x* and *y* have the same color.

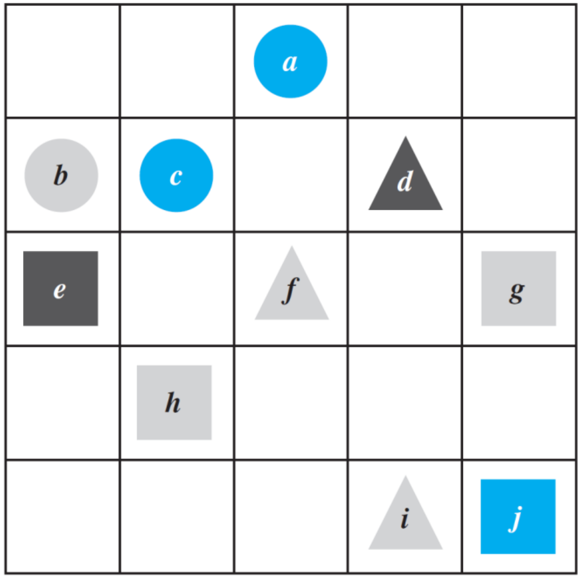


Figure 3.3.1

Negation:

∃ a square *x* such that ~(∃ a circle *y* such that *x* and *y* have the same color)

➔ ∃ a square *x* such that ∀ circles *y*, *x* and *y* do not have the same color.

TRUE (Square *e* is black and no circle is black).

Refer to the Tarski's world of Figure 3.3.1 again.

Write a negation for each of the following statements, and determine which is true, the given statement or its negation.

b. There is a triangle *x* such that for all squares *y*, *x* is to the right of *y*.

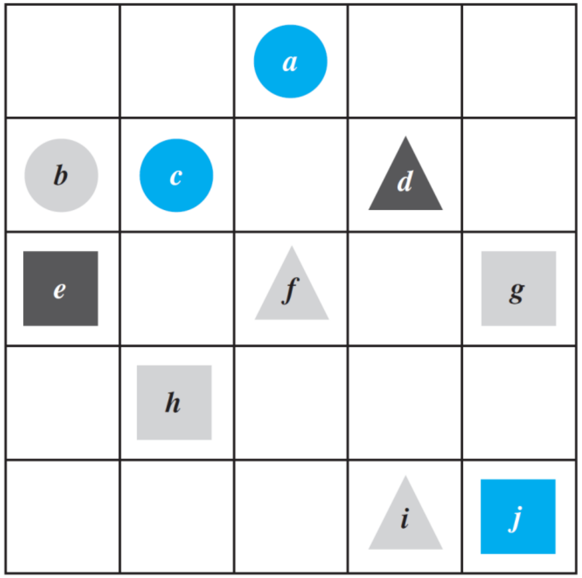


Figure 3.3.1

Negation:

∀ triangles *x*, ~(∀ squares *y*, *x* is to the right of *y*)

➔ ∀ triangles *x*, ∃ a square *y* such that *x* is not to the right of *y*.

TRUE (No matter what triangle is chosen, it is not to the right of square *g* or square *j*).

3.3.5. Order of Quantifiers

∀ people x, ∃ a person y such that x loves y.

∃ a person y such that ∀ people x, x loves y.

Except for the order of the quantifiers, these statements are identical.

Given any person, it is possible to find someone whom that person loves.

They are not logically equivalent!

There is one amazing individual who is loved by all people!

In a statement containing both ∀ and ∃, changing the order of the quantifiers usually changes the meaning of the statement.

However, if one quantifier immediately follows another quantifier ***of the same type***, then the order of the quantifiers does not affect the meaning.

Examples:

- ∀*x* ∀*y* is equivalent to ∀*y* ∀*x* (likewise for ∃)
- ∀*x* ∀*y* *may* be written as ∀*x*,*y* (likewise for ∃)

Refer to the Tarski's world of Figure 3.3.1. What are the truth values of the following two statements?

a. For every square *x*, there is a triangle *y* such that *x* and *y* have different colors.

TRUE

b. There exists a triangle *y* such that for every square *x*, *x* and *y* have different colors.

FALSE

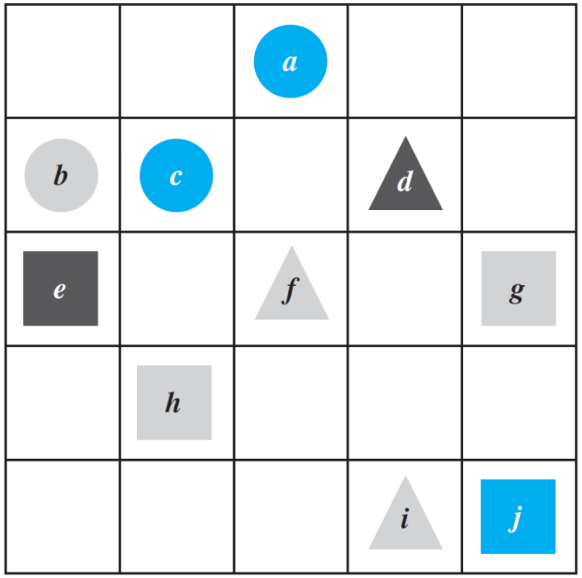


Figure 3.3.1

3.3.6. Formal Logical Notation

In some areas of computer science, logical statements are expressed in purely symbolic notation.

The notation involves using predicates to describe all properties of variables and omitting the words *such as* in existential statements.

"$\forall x \in D, P(x)$" written as $\forall x\ (x \in D \rightarrow P(x))$

"$\exists x \in D$ such that $P(x)$" written as $\exists x\ (x \in D \wedge P(x))$

We will follow this way of writing.

Example:

- Tarski's world.
- Let the common domain *D* of all variables be the set of all the objects in the Tarski's world.

Triangle(x): "x is a triangle"
Circle(x): "x is a circle"
Square(x): "x is a square"

Blue(x): "x is blue"
Gray(x): "x is gray"
Black(x): "x is black"

RightOf(x,y): "x is to the right of y"
Above(x,y): "x is above y"
SameColorAs(x,y): "x has the same color as y"
$x = y$: "x is equal to y"

Use formal, logical notation to write the following statements, and write a formal negation for each statement.

a. For all circles *x*, *x* is above *f*.

Statement: $\forall x\,(\text{Circle}(x) \to \text{Above}(x, f))$

Negation: $\sim(\forall x\,(\text{Circle}(x) \to \text{Above}(x, f)))$

$\equiv \exists x \sim(\text{Circle}(x) \to \text{Above}(x, f)) \equiv \exists x\,(\text{Circle}(x) \wedge \sim\text{Above}(x, f))$

b. There is a square *x* such that *x* is black.

Statement: $\exists x\,(\text{Square}(x) \wedge \text{Black}(x))$

Negation: $\sim(\exists x\,(\text{Square}(x) \wedge \text{Black}(x)))$

$\equiv \forall x \sim(\text{Square}(x) \wedge \text{Black}(x)) \equiv \forall x\,(\sim\text{Square}(x) \vee \sim\text{Black}(x))$

Use formal, logical notation to write the following statements, and write a formal negation for each statement.

c. For all circles *x*, there is a square *y* such that *x* and *y* have the same color.

Statement: $\forall x(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$

Negation: $\sim(\forall x(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

$\equiv \exists x \sim(\text{Circle}(x) \rightarrow \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$

$\equiv \exists x\, (\text{Circle}(x) \wedge \sim(\exists y\, (\text{Square}(y) \wedge \text{SameColor}(x, y))))$

$\equiv \exists x\, (\text{Circle}(x) \wedge \forall y\, (\sim(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

$\equiv \exists x\, (\text{Circle}(x) \wedge \forall y\, (\sim\text{Square}(y) \vee \sim\text{SameColor}(x, y)))$

Use formal, logical notation to write the following statements, and write a formal negation for each statement.

d. There is a square *x* such that for all triangles *y*, *x* is to right of *y*.

Statement: $\exists x\,(\text{Square}(x) \wedge \forall y\,(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y)))$

Negation: $\sim(\exists x\,(\text{Square}(x) \wedge \forall y\,(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$

$\equiv \forall x \sim(\text{Square}(x) \wedge \forall y\,(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y)))$

$\equiv \forall x\,(\sim\text{Square}(x) \vee \sim(\forall y\,(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$

$\equiv \forall x\,(\sim\text{Square}(x) \vee \exists y\,(\sim(\text{Triangle}(y) \rightarrow \text{RightOf}(x, y))))$

$\equiv \forall x\,(\sim \text{Square}(x) \vee \exists y\,(\text{Triangle}(y) \wedge \sim\text{RightOf}(x, y)))$

Formal logical notation is used in branches of computer science such as *artificial intelligence*, *program verification*, and *automata theory and formal languages*.

Taken together, the symbols for quantifiers, variables, predicates, and logical connectives make up what is known as the **language of first-order logic**.

Even though this language is simpler in many respects than the language we use every day, learning it requires the same kind of practice needed to acquire any foreign language.

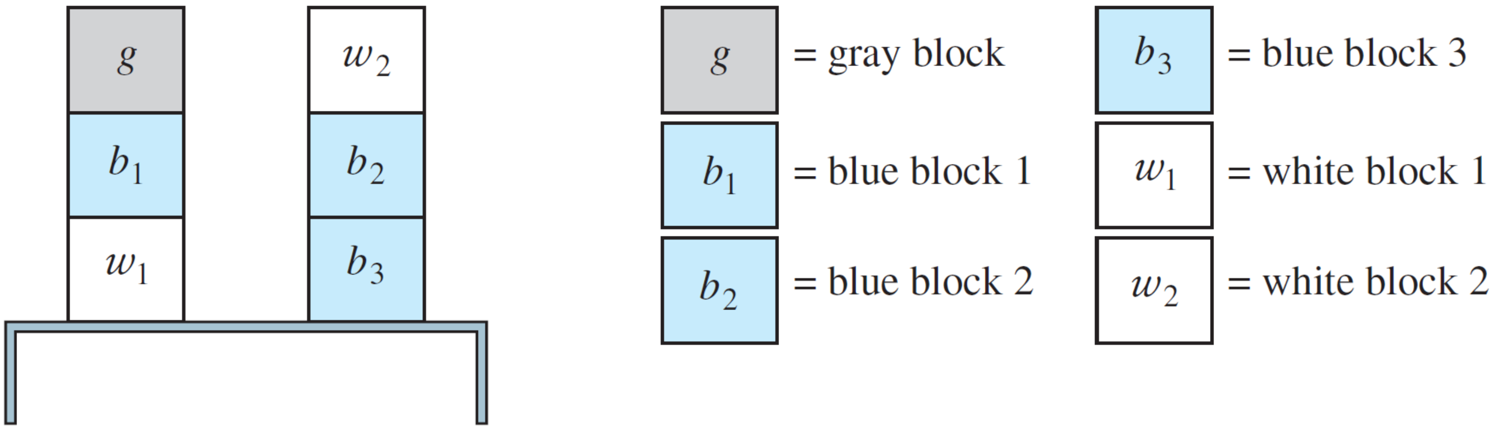
3.3.7. Prolog (Only for your reading)

The programming language **Prolog** (short for *pro*gramming in *log*ic) was developed in France in the 1970s by A. Colmerauer and P. Roussel to help programmers working in the field of artificial intelligence.

A simple Prolog program consists of a set of statements describing some situation together with questions about the situation. Built into the language are search and inference techniques needed to answer the questions by deriving the answers from the given statements.

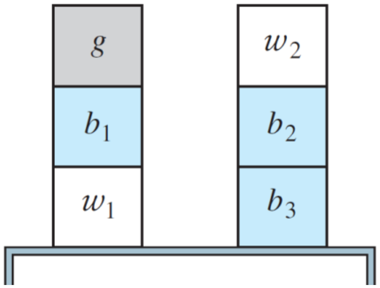
This frees the programmer from the necessity of having to write separate programs to answer each type of question.

Consider the following picture, which shows colored blocks stacked on a table.



The following are statements in Prolog that describe this picture and ask two questions about it.

Prolog: A Prolog Program



isabove(g, b_1) color(g, gray) color(b_3, blue)

isabove(b_1, w_1) color(b_1, blue) color(w_1, white)

isabove(w_2, b_2) color(b_2, blue) color(w_2, white)

isabove(b_2, b_3) isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)

?color(b_1, blue) ?isabove(X, w_1)

The statements "isabove(g, b_1)" and "color(g, gray)" are to be interpreted as "g is above b_1" and "g is colored gray".

The statement "isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)" is to be interpreted as "For all X, Y, and Z, if X is above Y and Y is above Z, then X is above Z."

The program statement

?color(b_1, blue)

is a question asking whether block b_1 is colored blue.

Prolog answers this by writing

Yes

The program statement

?isabove(X, w_1)

is a question asking for which blocks X the predicate "X is above w_1" is true..

Prolog answers this by giving a list of all such blocks. In this case, the answer is

$X = b_1, X = g$.

Infer the solution $X = g$ from the following statements:

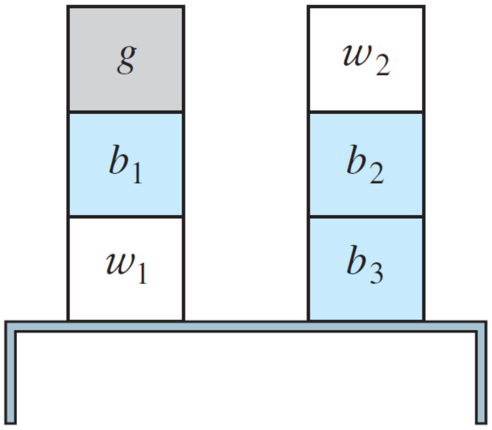
- isabove(g, b_1)
- isabove(b_1, w_1)
- isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)

Write the answers Prolog would give if the following questions were added to the program above.

a. ?isabove(b_2, w_1) "No"

b. ?color(w_1, X) "X = white"

c. ?color(X, blue) "$X = b_1$", "$X = b_2$", "$X = b_3$"



isabove(g, b_1)	color(g, gray)	color(b_3, blue)
isabove(b_1, w_1)	color(b_1, blue)	color(w_1, white)
isabove(w_2, b_2)	color(b_2, blue)	color(w_2, white)
isabove(b_2, b_3)	isabove(X, Z) if isabove(X, Y) and isabove(Y, Z)	

3.4 Arguments with Quantified Statements

3.4.1. Universal Instantiation

The rule of universal instantiation:

> If some property is true of *everything* in the set, then it is true of *any particular* thing in the set.

Universal instantiation is the fundamental tool of deductive reasoning.

3.4.2. Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called universal modus ponens.

Universal Modus Ponens

Formal version	*Informal version*
$\forall x\,(P(x) \rightarrow Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
• $Q(a)$.	• a makes $Q(x)$ true.

Recognizing Universal Modus Ponens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

> If an integer is even, then its square is even.
> *k* is a particular integer that is even.
> • k^2 is even.

Solution:

Premise: $\forall x$, if x is an even integer then x^2 is even.

Let $E(x)$ be "x is an even integer", let $S(x)$ be "x^2 is even", and let k stand for a particular integer that is even.

$\forall x\ (E(x) \to S(x))$.
$E(k)$, for a particular k.
• $S(k)$.

This argument has the form of universal modus ponens and is therefore valid.

3.4.3. Use of Universal Modus Ponens in a Proof

Proof: The sum of any two even integers is even.

$\forall$integers x, x is even iff $\exists$ an integer k such that $x = 2k$.

Suppose m and n are particular but arbitrarily chosen even integers, then $m = 2r$ for some integer r(1), and $n = 2s$ for some integer s(2).

Hence

$$m + n = 2r + 2s = 2(r + s) \quad (3)$$

Now $(r + s)$ is an integer(4), and so $2(r + s)$ is even(5).

Thus $m + n$ is even.

How universal modus ponens is used in the proof.

Suppose *m* and *n* are particular but arbitrarily chosen even integers, then $m = 2r$ for some integer r(1), and $n = 2s$ for some integer s(2).

(1) If an integer is even, then it equals twice some integer.
m is a particular even integer.
• *m* equals twice some integer *r*.

(2) Similar to (1).

How universal modus ponens is used in the proof.

Hence

$$m + n = 2r + 2s = 2(r + s)$$ (3)

(3) If a quantity is an integer, then it is a real number.

r and *s* are particular integers.

- *r* and *s* are real numbers.

We want to show that if r and s are integers, then they are real numbers, so that we can apply distributive law below (Appendix A, F3), which are meant for real numbers.

For all *a*, *b*, and *c*, if *a*, *b*, and *c* are real numbers, then $ab + ac = a(b + c)$.

2, *r*, and *s* are particular real numbers.

- $2r + 2s = 2(r + s)$.

How universal modus ponens is used in the proof.

Now $(r + s)$ is an integer[(4)], and so $2(r + s)$ is even[(5)].

Thus $m + n$ is even.

(4) For all u and v, if u and v are integers, then $(u + v)$ is an integer.

r and s are two particular integers.

- $(r + s)$ is an integer.

(5) If a number equals twice some integer, then that number is even.

$2(r + s)$ equals twice the integer $(r + s)$.

- $2(r + s)$ is even.

3.4.4. Universal Modus Tollens

Another crucially important rule of inference is universal modus tollens. Its validity results from combining universal instantiation with modus tollens.

Universal modus tollens is the heart of proof of contradiction.

Universal Modus Tollens

Formal version	*Informal version*
$\forall x, (P(x) \to Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$ for a particular a.	a does not make $Q(x)$ true.
• $\sim P(a)$.	• a does not makes $P(x)$ true.

Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All human beings are mortal.
Zeus is not mortal.
• Zeus is not human.

Solution:

Premise: $\forall x$, if x is human then x is mortal.

Let $H(x)$ be "x is human", let $M(x)$ be "x is mortal", and let Z stand for Zeus.

$\forall x\,(H(x) \to M(x))$.
$\sim M(Z)$.
• $\sim H(Z)$.

This argument has the form of universal modus tollens and is therefore valid.

3.4.5. Proving Validity of Arguments with Quantified Statements

The intuitive definition of validity for arguments with quantified statements is the same as for arguments with compound statements.

An argument is valid if, and only if, the truth of its conclusion follows *necessarily* from the truth of its premises.

Definition 3.4.1 (Valid Argument Form)

To say that **an argument form is valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

An **argument is called valid** if, and only if, its form is valid.

3.4.6. Using Diagrams to Test for Validity

Consider the statement: All integers are rational numbers.

$\forall$integers *n*, *n* is a rational number.

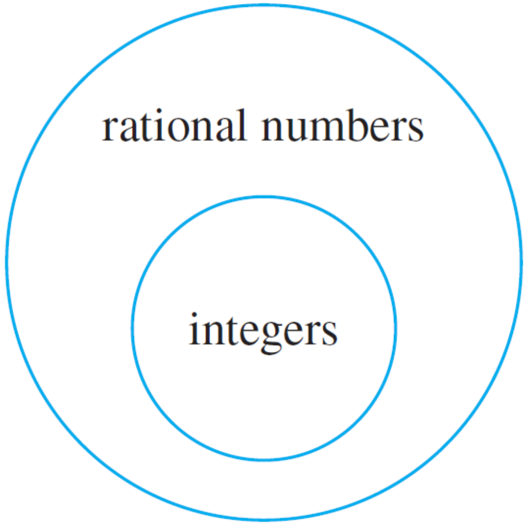


Figure 3.4.1

Use a diagram to show the invalidity of the following argument:

All human beings are mortal.
Felix is mortal.
● Felix is a human being.

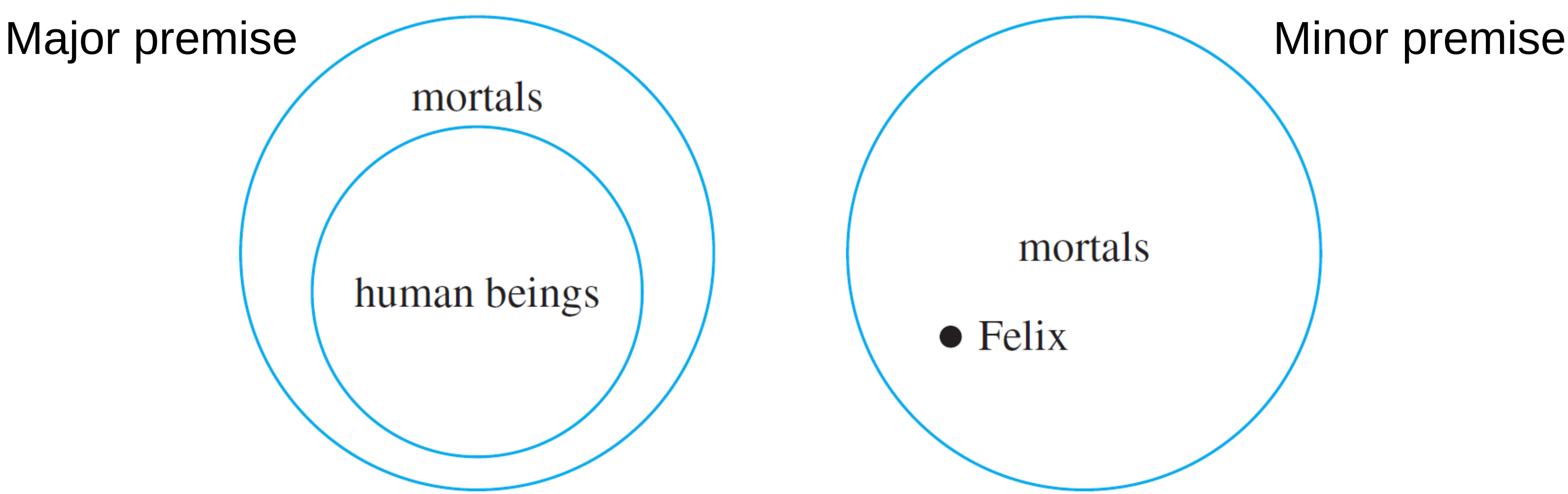


Figure 3.4.4

Use a diagram to show the invalidity of the following argument:

All human beings are mortal.
Felix is mortal.
∴ Felix is a human being.

Hence, argument is invalid.

Conclusion is true.

Conclusion is false.

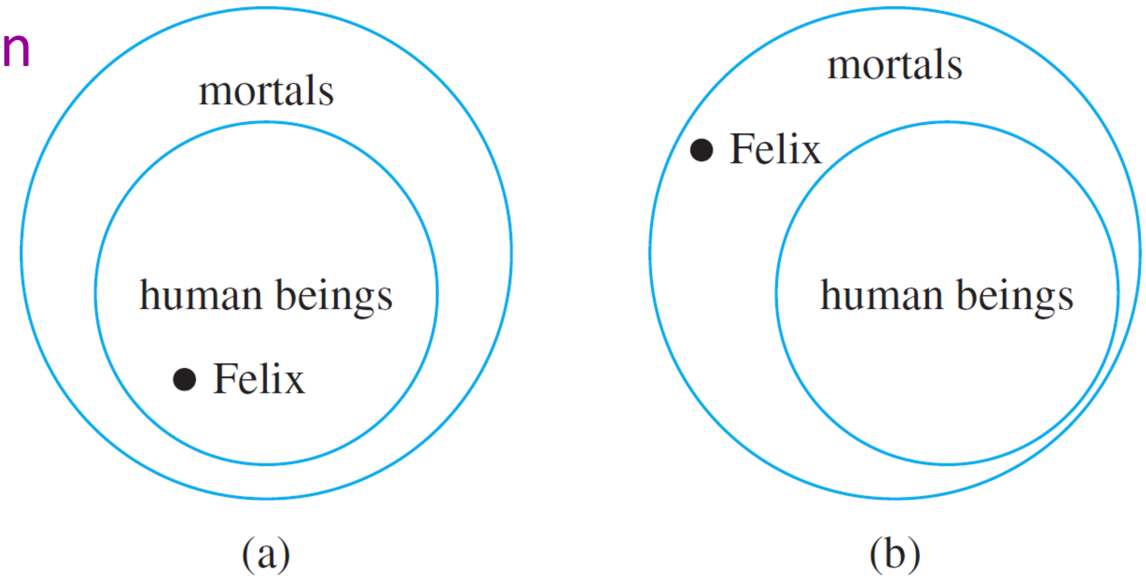


Figure 3.4.5

The argument of previous example would be valid if the major premise were replaced by its converse. But since a universal conditional statement is not logically equivalent to its converse, such a replacement cannot, in general, be made.

We say that this argument exhibit the converse error.

Converse Error (Quantified Form)

Formal version	*Informal version*
$\forall x\ (P(x) \rightarrow Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
• $P(a)$.	• a makes $P(x)$ true.

The following form of argument would be valid if a conditional statement were logically equivalent to its inverse. But it is not, and the argument form is invalid.

We say that this argument exhibit the inverse error.

Inverse Error (Quantified Form)

Formal version	*Informal version*
$\forall x\,(P(x) \to Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$ for a particular a.	a does not make $P(x)$ true.
• $\sim Q(a)$.	• a does not make $Q(x)$ true.

An Argument with "No"

Use diagrams to test the following argument for validity:

No polynomial functions have horizontal asymptotes.
This function has a horizontal asymptote.
• This function is not a polynomial function.

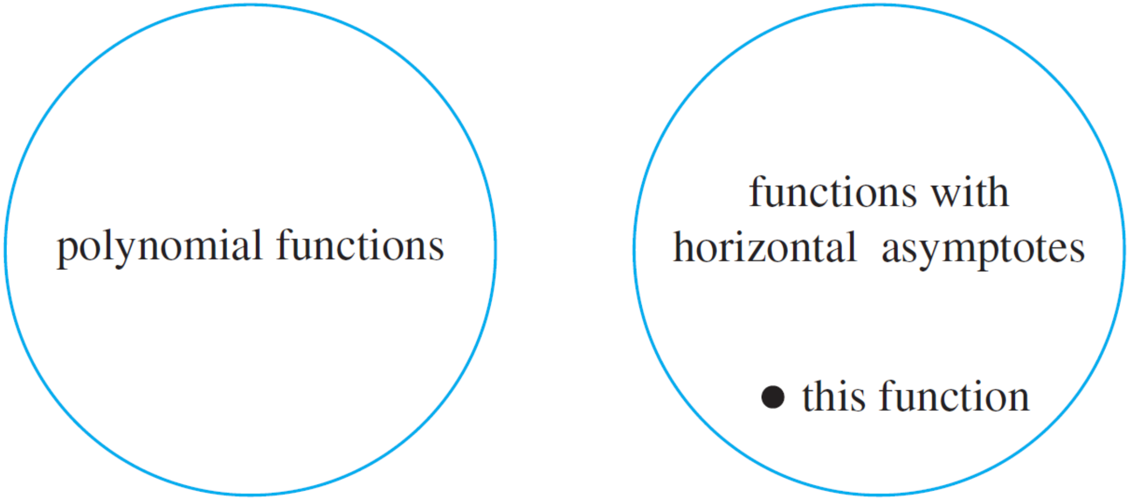


Figure 3.4.6

Hence argument is valid.

No polynomial functions have horizontal asymptotes.
This function has a horizontal asymptote.
• This function is not a polynomial function.

Alternatively, transform the first statement into:

∀*x*, if *x* is a polynomial function, then *x* does not have a horizontal asymptote.

Then the argument has the form:

$\forall x\ (P(x) \rightarrow Q(x))$.
$\sim Q(a)$, for a particular a.
• $\sim P(a)$.

This is valid by universal modus tollens.

3.4.7. Creating Additional Forms of Argument

We have seen:

Modus ponens + Universal instantiation ➔ Universal modus ponens

Modus tollens + Universal instantiation ➔ Universal modus tollens

In the same way, additional forms of arguments involving universally quantified statements can be obtained by combining universal instantiation with other of the valid argument forms discussed earlier.

Consider the following argument:

$$p \to q$$
$$q \to r$$
$$\bullet\ p \to r$$

This can be combined with universal instantiation to obtain a valid argument form.

Universal Transitivity

Formal version	*Informal version*
$\forall x\,(P(x) \to Q(x))$.	Any x that makes $P(x)$ true makes $Q(x)$ true.
$\forall x\,(Q(x) \to R(x))$.	Any x that makes $Q(x)$ true makes $R(x)$ true.
• $\forall x\,(P(x) \to R(x))$.	• Any x that makes $P(x)$ true makes $R(x)$ true.

Consider the Tarski's world:

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

1. All the triangles are blue.
2. If an object is to the right of all the squares, then it is above all the circles.
3. If an object is not to the right of all the squares, then it is not blue.

- All the triangles are above all the circles.

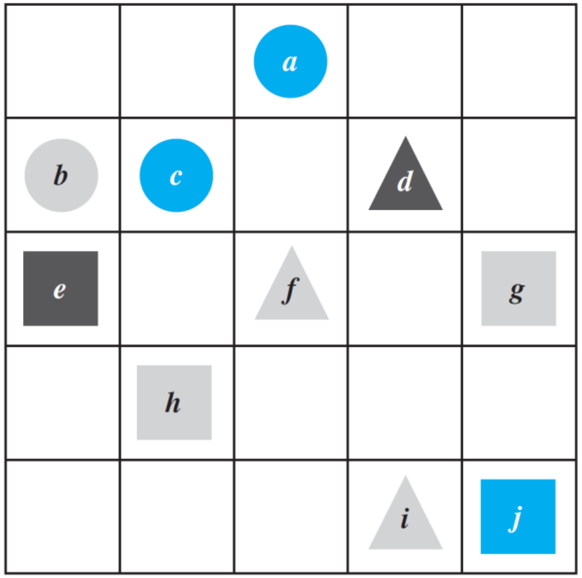


Figure 3.3.1

Evaluating an Argument for Tarski's World

Consider the Tarski's world:

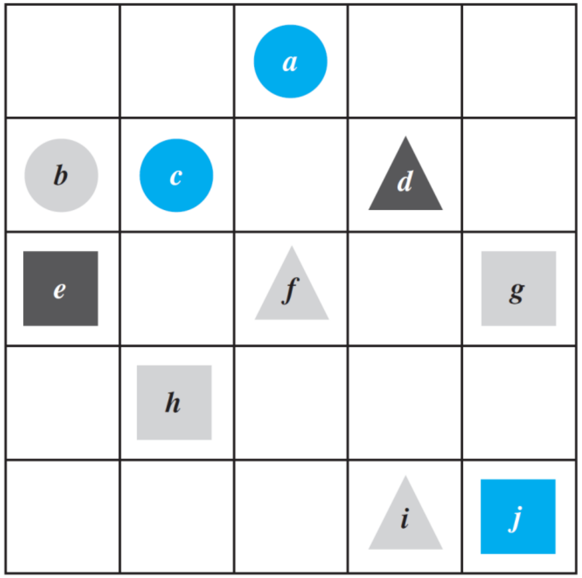


Figure 3.3.1

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

Step 1:

1. ∀*x*, if *x* is a triangle, then *x* is blue.
2. ∀*x*, if *x* is to the right of all the squares, then *x* is above all the circles.
3. ∀*x*, if *x* is not to the right of all the squares, then *x* is not blue.

- ∀*x*, if *x* is a triangle, then *x* is above all the circles.

Should be same as hypothesis of the first premise.

Should be same as conclusion of the last premise.

Consider the Tarski's world:

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

Step 2:

1. ∀*x*, if *x* is a triangle, then *x* is blue.
2. ∀*x*, if *x* is not to the right of all the squares, then *x* is not blue.
3. ∀*x*, if *x* is to the right of all the squares, then *x* is above all the circles.

- ∀*x*, if *x* is a triangle, then *x* is above all the circles.

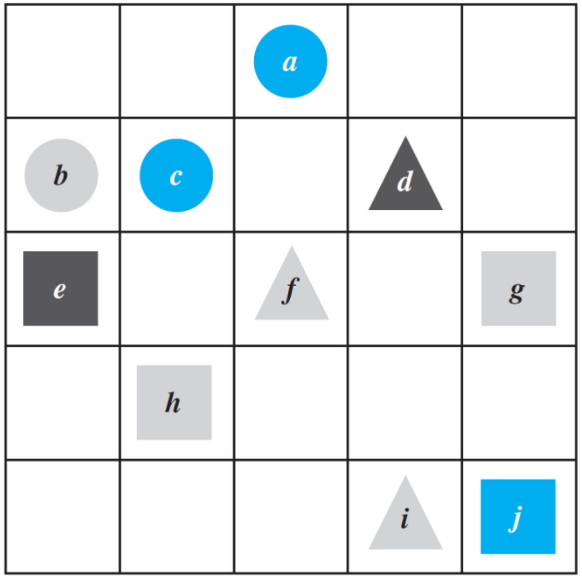


Figure 3.3.1

Rewrite it in contrapositive form.

Evaluating an Argument for Tarski's World

Consider the Tarski's world:

Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises

Step 3:

1. ∀*x*, if *x* is a triangle, then *x* is blue.
2. ∀*x*, if *x* is blue, then *x* is to the right of all the squares.
3. ∀*x*, if *x* is to the right of all the squares, then *x* is above all the circles.

- ∀*x*, if *x* is a triangle, then *x* is above all the circles.

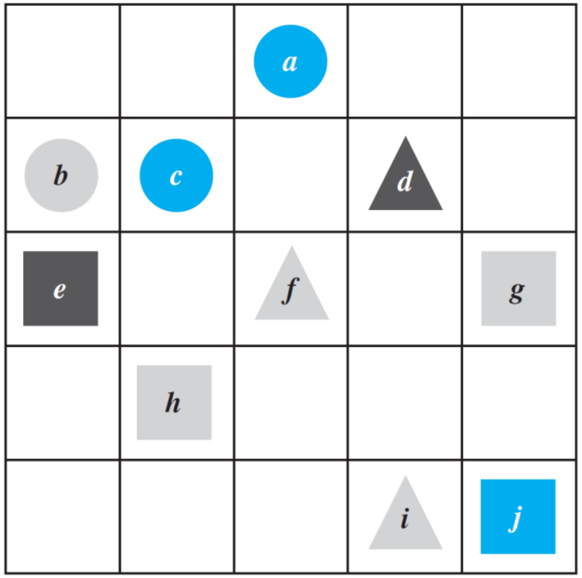


Figure 3.3.1

3.4.8. Rules of Inference for Quantified Statements

Rule of Inference	Name
$\forall x \in D\ P(x)$ $\therefore P(a)$ if $a \in D$	Universal instantiation
$P(a)$ for every $a \in D$ $\therefore \forall x \in D\ P(x)$	Universal generalization
$\exists x \in D\ P(x)$ $\therefore P(a)$ for some $a \in D$	Existential instantiation
$P(a)$ for some $a \in D$ $\therefore \exists x \in D\ P(x)$	Existential generalization

3.4.9. Remark on the Converse and Inverse Errors

Only for reading.

A variation of the converse error is a very useful reasoning tool, provided that it is used with caution.

It is the type of reasoning that is used by doctors to make medical diagnoses and by auto mechanics to repair cars.

It is the type of reasoning used to generate explanations for phenomena. It goes like this: If a statement of the form

$$\text{For all } x\ (P(x) \to Q(x))$$

is true, and if

$$Q(a) \text{ is true, for a particular } a,$$

then check out the statement $P(a)$; it just might be true.

Only for reading.

For instance, suppose a doctor knows that

> For all *x*, if *x* has pneumonia, then *x* has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

And suppose the doctor also knows that

> John has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

On the basis of these data, the doctor concludes that a diagnosis of pneumonia is a strong possibility, though not a certainty.

This form of reasoning has been named abduction by researchers working in artificial intelligence.

END OF FILE