3. The Logic of Quantified Statements (aka Predicate Logic) Summary

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Summary

3.1 Predicates and Quantified Statements I

- Predicate; domain; truth set
- Universal quantifier ∀, existential quantifier ∃
- Universal conditional statements; Implicit quantification

3.2 Predicates and Quantified Statements II

- Negation of quantified statements; negation of universal conditional statements
- Vacuous truth of universal statements
- Variants of universal conditional statements (contrapositive, converse, inverse)
- Necessary and sufficient conditions, only if

3.3 Statements with Multiple Quantifiers

- Negations of multiply-quantified statements; order of quantifiers
- Prolog

3.4 Arguments with Quantified Statements

- Universal instantiation; universal modus ponens; universal modus tollens

Summary

3.1 Predicates and Quantified Statements I

Definition 3.1.1 (Predicate)

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Definition 3.1.2 (Truth set)

If $P(x)$ is a predicate and x has domain D, the **truth set** is the set of all elements of D that make $P(x)$ true when they are substituted for x.
The truth set of $P(x)$ is denoted $\{x \in D \mid P(x)\}$.

Summary

Definition 3.1.3 (Universal Statement)

Let $Q(x)$ be a predicate and D the domain of x.

A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$".

- It is defined to be true iff $Q(x)$ is true for every x in D.
- It is defined to be false iff $Q(x)$ is false for at least one x in D.

A value for x for which $Q(x)$ is false is called a **counterexample.**

Definition 3.1.4 (Existential Statement)

Let $Q(x)$ be a predicate and D the domain of x.

An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$".

- It is defined to be true iff $Q(x)$ is true for at least one x in D.
- It is defined to be false iff $Q(x)$ is false for all x in D.

$\exists!$ is the **uniqueness quantifier symbol**. It means "there exists a unique" or "there is one and only one".

Summary

Theorem 3.2.1 Negation of a Universal Statement

The **negation** of a statement of the form

$$\forall x \in D, P(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D \text{ such that } \sim P(x)$$

Symbolically,

$$\sim(\forall x \in D, P(x)) \equiv \exists x \in D \text{ such that } \sim P(x)$$

Theorem 3.2.2 Negation of an Existential Statement

The **negation** of a statement of the form

$$\exists x \in D \text{ such that } P(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D, \sim P(x)$$

Symbolically,

$$\sim(\exists x \in D \text{ such that } P(x)) \equiv \forall x \in D, \sim P(x)$$

Summary

Definition 3.2.1 (Contrapositive, converse, inverse)

Consider a statement of the form: $\forall x \in D\ (P(x) \rightarrow Q(x))$.

1. Its **contrapositive** is: $\forall x \in D\ (\sim Q(x) \rightarrow \sim P(x))$.
2. Its **converse** is: $\forall x \in D\ (Q(x) \rightarrow P(x))$.
3. Its **inverse** is: $\forall x \in D\ (\sim P(x) \rightarrow \sim Q(x))$.

Definition 3.2.2 (Necessary and Sufficient conditions, Only if)

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x\ (r(x) \rightarrow s(x))$".
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x\ (\sim r(x) \rightarrow \sim s(x))$" or, equivalently, "$\forall x\ (s(x) \rightarrow r(x))$".
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x\ (\sim s(x) \rightarrow \sim r(x))$" or, equivalently, "$\forall x\ (r(x) \rightarrow s(x))$" .

Summary

3.4 Arguments with Quantified Statements

Universal Modus Ponens

Formal version	*Informal version*
$\forall x\,(P(x) \to Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
• $Q(a)$.	• a makes $Q(x)$ true.

Universal Modus Tollens

Formal version	*Informal version*
$\forall x\,(P(x) \to Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$ for a particular a.	a does not make $Q(x)$ true.
• $\sim P(a)$.	• a does not makes $P(x)$ true.

Definition 3.4.1 (Valid Argument Form)

To say that **an argument form is valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

An **argument is called valid** if, and only if, its form is valid.

Summary

Converse Error (Quantified Form)

Formal version	*Informal version*
$\forall x\,(P(x) \to Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
• $P(a)$.	• a makes $P(x)$ true.

Inverse Error (Quantified Form)

Formal version	*Informal version*
$\forall x\,(P(x) \to Q(x))$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$ for a particular a.	a does not make $P(x)$ true.
• $\sim Q(a)$.	• a does not make $Q(x)$ true.

Universal Transitivity

Formal version	*Informal version*
$\forall x\,(P(x) \to Q(x))$.	Any x that makes $P(x)$ true makes $Q(x)$ true.
$\forall x\,(Q(x) \to R(x))$.	Any x that makes $Q(x)$ true makes $R(x)$ true.
• $\forall x\,(P(x) \to R(x))$.	• Any x that makes $P(x)$ true makes $R(x)$ true.

Summary

Rule of Inference for quantified statements	Name
$\forall x \in D\ P(x)$ $\therefore P(a)$ if $a \in D$	Universal instantiation
$P(a)$ for every $a \in D$ $\therefore \forall x \in D\ P(x)$	Universal generalization
$\exists x \in D\ P(x)$ $\therefore P(a)$ for some $a \in D$	Existential instantiation
$P(a)$ for some $a \in D$ $\therefore \exists x \in D\ P(x)$	Existential generalization

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