# Lecture #10: Counting and Probability I Summary

Aaron Tan

AY2024/25 Semester 1

# Summary



This lecture is based on Epp's book chapter 9.
Hence, the section numbering is according to the book.

## 9.1 Introduction

- Random process, sample space, event and probability

## 9.2 Possibility Trees and the Multiplication Rule

- Possibility trees
- The multiplication/product rule
- Permutations

## 9.3 Counting Elements of Disjoint Sets

- The addition/sum rule
- The difference rule
- The inclusion/exclusion rule

## 9.4 The Pigeonhole Principle (PHP)

- Pigeonhole principle, general pigeonhole principle

Reference: Epp's Chapter 9 Counting and Probability

# Summary

## 9.1 Introduction

### Definitions

A **sample space** is the set of all possible outcomes of a random process or experiment.
An **event** is a subset of a sample space.

### Notation

For a finite set $A$, $|A|$ denotes the number of elements in $A$.

### Equally Likely Probability Formula

If $S$ is a finite sample space in which all outcomes are equally likely and $E$ is an event in $S$, then the **probability** of $E$, denoted $P(E)$, is

$$P(E) = \frac{\text{The number of outcomes in } E}{\text{The total number of outcomes in } S} = \frac{|E|}{|S|}$$

### Theorem 9.1.1 The Number of Elements in a List

If $m$ and $n$ are integers and $m \le n$, then there are $n - m + 1$ integers from $m$ to $n$ inclusive.

# Summary



**Theorem 9.2.1 The Multiplication/Product Rule**

If an operation consists of $k$ steps and

the first step can be performed in $n_1$ ways,

the second step can be performed in $n_2$ ways (regardless of how the first step was performed),

:

the $k^{th}$ step can be performed in $n_k$ ways (regardless of how the preceding steps were performed),

Then the entire operation can be performed in

$$n_1 \times n_2 \times n_3 \times \ldots \times n_k \text{ ways.}$$

### Theorem 9.2.2 Permutations

The number of permutations of a set with $n$ ($n \geq 1$) elements is ***n*!**

### Definition

An ***r*-permutation** of a set of $n$ elements is an ordered selection of $r$ elements taken from the set.

The number of $r$-permutations of a set of $n$ elements is denoted $\boldsymbol{P(n, r)}$.

### Theorem 9.2.3 $r$-permutations from a set of $n$ elements

If $n$ and $r$ are integers and $1 \leq r \leq n$, then the number of $r$-permutations of a set of $n$ elements is given by the formula

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$ first version

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!}$$ second version

# Summary



## Theorem 9.3.1 The Addition/Sum Rule

Suppose a finite set $A$ equals the union of $k$ distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then $|A| = |A_1| + |A_2| + \ldots + |A_k|$.

## Theorem 9.3.2 The Difference Rule

If $A$ is a finite set and $B \subseteq A$, then $|A \setminus B| = |A| - |B|$.

## Formula for the Probability of the Complement of an Event

If $S$ is a finite sample space and $A$ is an event in $S$, then $P(\bar{A}) = 1 - P(A)$.

## Theorem 9.3.3 The Inclusion/Exclusion Rule for 2 or 3 Sets

If $A$, $B$, and $C$ are any finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

## Pigeonhole Principle (PHP)

A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least 2 elements in the domain that have the same image in the co-domain.

## Generalized Pigeonhole Principle

For any function $f$ from a finite set $X$ with $n$ elements to a finite set $Y$ with $m$ elements and for any positive integer $k$, if $k < n/m$, then there is some $y \in Y$ such that $y$ is the image of at least $k + 1$ distinct elements of $X$.

## Generalized Pigeonhole Principle (Contrapositive Form)

For any function $f$ from a finite set $X$ with $n$ elements to a finite set $Y$ with $m$ elements and for any positive integer $k$, if for each $y \in Y$, $f^{-1}(\{y\})$ has at most $k$ elements, then $X$ has at most $km$ elements; in other words, $n \leq km$.

# END OF FILE