Lecture #12: Graphs
Summary

Aaron Tan

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10.1 Graphs: Definitions and Basic Properties

- Introduction, Basic Terminology
- Special Graphs
- The Concept of Degree

10.2 Trails, Paths, and Circuits

- Definitions
- Connectedness
- Euler Circuits and Hamiltonian Circuits

10.3 Matrix Representations of Graphs

- Matrices and Directed Graphs; Matrices and Undirected Graphs
- Matrix Multiplication
- Counting Walks of Length *N*

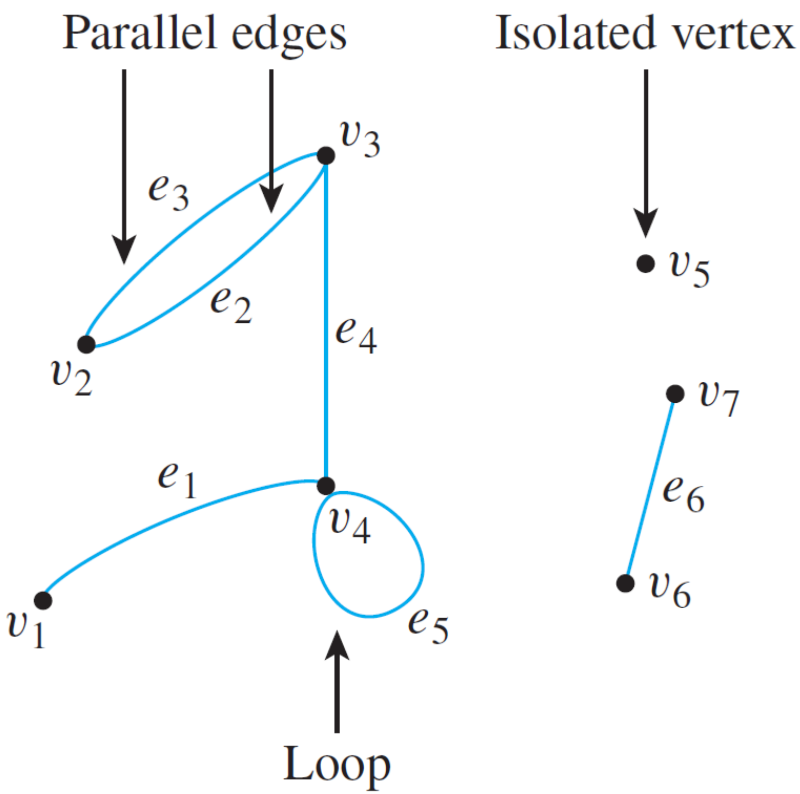
10.4 Isomorphisms of Graphs/Planar Graphs

- Definition of Graph Isomorphism
- Planar Graphs and Euler's Formula

Reference: Epp's Chapter 10 Graphs and Trees

An **undirected graph** $G = (V, E)$ consists of

- a set of vertices $V = \{v_1, v_2, \cdots, v_n\}$, and
- a set of (undirected) edges $E = \{e_1, e_2, \cdots, e_k\}$.
- An (undirected) edge e connecting v_i and v_j is denoted as $e = \{v_i, v_j\}$.



$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$
$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

$e_1 = \{v_1, v_4\}$
$e_2 = e_3 = \{v_2, v_3\}$
$e_4 = \{v_3, v_4\}$
$e_5 = \{v_4, v_4\}$
$e_6 = \{v_6, v_7\}$

Edges incident on v_4: e_1, e_4 and e_5.
Vertices adjacent to v_4: v_1, v_3 and v_4.
Edges adjacent to e_2: e_3 and e_4.

Definition: Undirected Graph

An undirected **graph** *G* consists of 2 finite sets: a nonempty set *V* of **vertices** and a set *E* of **edges**, where each (undirected) edge is associated with a set consisting of either one or two vertices called its **endpoints**.

An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent vertices**; and a vertex that is an endpoint of a loop is said to be **adjacent to itself**.

An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent edges**.

We write $e = \{v, w\}$ for an undirected edge *e* incident on vertices *v* and *w*.

Definition: Directed Graph

A **directed graph**, or **digraph**, *G*, consists of 2 finite sets: a nonempty set *V* of **vertices** and a set *E* of **directed edges**, where each (directed) edge is associated with an ordered pair of vertices called its **endpoints**.

We write $e = (v, w)$ for a directed edge *e* from vertex v to vertex w.

Summary

Definition: Simple Graph

A **simple graph** is an undirected graph that does not have any loops or parallel edges. (That is, there is at most one edge between each pair of distinct vertices.)

Definition: Complete Graph

A **complete graph** on ***n* vertices**, $n > 0$, denoted $\boldsymbol{K_n}$, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices.

Definition: Bipartite Graph

A **bipartite graph** (or bigraph) is a simple graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V.

Definition: Complete Bipartite Graph

A **complete bipartite graph** is a bipartite graph on two disjoint sets U and V such that every vertex in U connects to every vertex in V.

If $|U| = m$ and $|V| = n$, the complete bipartite graph is denoted as $K_{m,n}$.

Summary

Definition: Subgraph of a Graph

A graph *H* is said to be a **subgraph** of graph *G* iff every vertex in *H* is also a vertex in *H*, every edge in *H* is also an edge in *G*, and every edge in *H* has the same endpoints as it has in *G*.

Definition: Degree of a Vertex and Total Degree of a Graph

Let *G* be a graph and *v* a vertex of *G*. The **degree** of *v*, denoted **deg(*v*)**, equals the number of edges that are incident on *v*, with an edge that is a loop counted twice.

The **total degree of *G*** is the sum of the degrees of all the vertices of *G*.

Theorem 10.1.1 The Handshake Theorem

If the vertices of *G* are $v_1, v_2, \ldots, v_n$, where $n \geq 0$, then the total degree of *G* $= \deg(v_1) + \deg(v_2) + \ldots + \deg(v_n) = 2 \times$ (the number of edges of *G*).

Corollary 10.1.2

The total degree of a graph is even.

Proposition 10.1.3

In any graph there are an even number of vertices of odd degree.

Definition: Indegree and outdegree of a Vertex of a Directed Graph

Let $G=(V,E)$ be a directed graph and v a vertex of G. The **indegree** of v, denoted **$\deg^-(v)$**, is the number of directed edges that end at v. The **outdegree** of v, denoted **$\deg^+(v)$**, is the number of directed edges that originate from v.

Note that

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|$$

Definitions

Let G be a graph, and let v and w be vertices of G.

A **walk from *v* to *w*** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form $v_0 e_1 v_1 e_2 \ldots v_{n-1} e_n v_n$, where the v's represent vertices, the e's represent edges, $v_0=v$, $v_n=w$, and for all $i \in \{1, 2, \ldots, n\}$, v_{i-1} and v_i are the endpoints of e_i. The number of edges, n, is the **length** of the walk.

The **trivial walk** from v to v consists of the single vertex v.

A **trail from *v* to *w*** is a walk from v to w that does not contain a repeated edge.

A **path from *v* to *w*** is a trail that does not contain a repeated vertex.

A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** (or **cycle**) is a closed walk of length at least 3 that does not contain a repeated edge.

A **simple circuit** (or **simple cycle**) is a circuit that does not have any other repeated vertex except the first and last.

An undirected graph is **cyclic** if it contains a loop or a cycle; otherwise, it is **acyclic**.

Summary

Definition: Connectedness

Two vertices *v* and *w* of a graph *G* are **connected** iff there is a walk from *v* to *w*.

The graph *G* is connected iff given *any* two vertices *v* and *w* in *G*, there is a walk from *v* to *w*. Symbolically, *G* is connected iff $\forall$ vertices $v, w \in V(G)$, $\exists$ a walk from *v* to *w*.

Lemma 10.2.1

Let *G* be a graph.

a. If *G* is connected, then any two distinct vertices of *G* can be connected by a path.
b. If vertices *v* and *w* are part of a circuit in *G* and one edge is removed from the circuit, then there still exists a trail from *v* to *w* in *G*.
c. If *G* is connected and *G* contains a circuit, then an edge of the circuit can be removed without disconnecting *G*.

Definition: Connected Component

A graph *H* is a **connected component** of a graph *G* iff

1. The graph *H* is a subgraph of *G*;
2. The graph *H* is connected; and
3. No connected subgraph of *G* has *H* as a subgraph and contains vertices or edges that are not in *H*.

Summary

Definitions: Euler Circuit and Eulerian Graph

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G.

An **Eulerian graph** is a graph that contains an Euler circuit.

Theorem 10.2.2

If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Contrapositive Version of Theorem 10.2.2

If some vertex of a graph has odd degree, then the graph doesn't have an Euler circuit.

Theorem 10.2.3

If a graph G is connected and the degree of every vertex of G is a positive even integer, then G has an Euler circuit.

Theorem 10.2.4

A graph G has an Euler circuit iff G is connected and every vertex of G has positive even degree.

Summary

Definition: Euler Trail

Let *G* be a graph, and let *v* and *w* be two distinct vertices of *G*. An **Euler trail/path from *v* to *w*** is a sequence of adjacent edges and vertices that starts at *v*, ends at *w*, passes through every vertex of *G* at least once, and traverses every edge of *G* exactly once.

Corollary 10.2.5

Let *G* be a graph, and let *v* and *w* be two distinct vertices of *G*. There is an Euler trail from *v* to *w* iff *G* is connected, *v* and *w* have odd degree, and all other vertices of *G* have positive even degree.

Definitions: Hamiltonian Circuit and Hamiltonian Graph

Given a graph *G*, a **Hamiltonian circuit** for *G* is a simple circuit that includes every vertex of *G*. (That is, every vertex appears exactly once, except for the first and the last, which are the same.)

A **Hamiltonian graph** (also called **Hamilton graph**) is a graph that contains a Hamiltonian circuit.

Proposition 10.2.6

If a graph *G* has a Hamiltonian circuit, then *G* has a subgraph *H* with the following properties:

1. *H* contains every vertex of *G*.
2. *H* is connected.
3. *H* has the same number of edges as vertices.
4. Every vertex of *H* has degree 2.

Definition: Adjacency Matrix of a Directed Graph

Let G be a directed graph with ordered vertices $v_1, v_2, \ldots v_n$. The **adjacency matrix of G** is the $n \times n$ matrix **A** = (a_{ij}) over the set of non-negative integers such that

$$a_{ij} = \text{the number of arrows from } v_i \text{ to } v_j \quad \text{for all } i, j = 1, 2, \ldots, n.$$

Definition: Adjacency Matrix of an Undirected Graph

Let G be an undirected graph with ordered vertices $v_1, v_2, \ldots v_n$. The **adjacency matrix of G** is the $n \times n$ matrix **A** = (a_{ij}) over the set of non-negative integers such that

$$a_{ij} = \text{the number of edges connecting } v_i \text{ and } v_j \quad \text{for all } i, j = 1, 2, \ldots, n.$$

Definition: Symmetric Matrix

An $n \times n$ square matrix A = (a_{ij}) is called **symmetric** iff for all $i, j = 1, 2, \ldots, n$,

$$a_{ij} = a_{ji}.$$

Definition: n^{th} Power of a Matrix

For any $n \times n$ matrix **A**, the **powers of A** are defined as follows:

$\mathbf{A}^0 = \mathbf{I}$ where **I** is the $n \times n$ identity matrix

$\mathbf{A}^n = \mathbf{A}\,\mathbf{A}^{n-1}$ for all integers $n \geq 1$

Theorem 10.3.2

If G is a graph with vertices $v_1, v_2, \ldots, v_m$ and **A** is the adjacency matrix of G, then for each positive integer n and for all integers $i, j = 1, 2, \ldots, m$,

the ij-th entry of $\mathbf{A}^n$ = the number of walks of length n from v_i to v_j.

Definition: Isomorphic Graph

Let $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ be two graphs.

***G* is isomorphic to *G*′**, denoted $G \cong G'$, if and only if there exist bijections $g: V_G \to V_{G'}$ and $h: E_G \to E_{G'}$ that preserve the edge-endpoint functions of G and G' in the sense that for all $v \in V_G$ and $e \in E_G$,

v is an endpoint of $e \Leftrightarrow g(v)$ is an endpoint of $h(e)$.

Alternative definition (for simple graphs)

Let $G = (V_G, E_G)$ and $G' = (V_{G'}, E_{G'})$ be two simple graphs.

***G* is isomorphic to *G*′** if and only if there exists a permutation $\pi: V_G \to V_{G'}$ such that $\{u, v\} \in E_G \Leftrightarrow \{\pi(u), \pi(v)\} \in E_{G'}$.

Theorem 10.4.1 Graph Isomorphism is an Equivalence Relation

Let *S* be a set of graphs and let $\cong$ be the relation of graph isomorphism on *S*.
Then $\cong$ is an equivalence relation on *S*.

Definition: Planar Graph

A **planar graph** is a graph that can be drawn on a (two-dimensional) plane without edges crossing.

Euler's Formula

For a connected planar simple graph $G = (V, E)$ with $e = |E|$ and $v = |V|$, if we let f be the number of faces, then

$$f = e - v + 2$$

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