NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING

MID-TERM TEST
AY2022/23 Semester 2

CS1231S — DISCRETE STRUCTURES

8 March 2023 Time Allowed: 1 hour 30 minutes

INSTRUCTIONS

1. This assessment paper contains **SEVENTEEN (17)** questions in **TWO (2)** parts and comprises **SEVEN (7)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions.
5. Write your answers only on the **ANSWER SHEET**. You may write in pen or pencil. You are to write within the space provided. No extra pages should be submitted.
6. The maximum mark of this assessment is 50.
7. Do not start writing or flip over this page until you are told to do so.

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions (Total: 24 marks)

Each multiple choice question (MCQ) is worth two marks and has exactly **one** correct answer.

1. Name the two venues where CS1231S lectures are conducted this semester.

   A. LT15 and LT19.

   B. ICube Auditorium and LT15.

   C. ICube Auditorium and LT19.

   D. Somewhere in NUS.

   E. There are no physical lectures this semester.

2. Given the following statements, pick the odd one out.

   A. $true \rightarrow false$

   B. $false \rightarrow true$

   C. $p \rightarrow p$

   D. $(p \wedge q) \rightarrow (p \vee q)$

   E $(p \vee q \vee {\sim}r) \wedge \big(s \vee {\sim}(p \vee q)\big) \rightarrow (s \vee {\sim}r)$

3. Given the following statement form:

$$(p \wedge {\sim}r \wedge q) \vee ({\sim}p \wedge r) \vee (r \wedge p) \vee \Big(\big(({\sim}p \wedge q) \vee s\big) \wedge r\Big)$$

   Which of the following is logically equivalent to the above?

   A. r

   B. $r \wedge (p \vee q)$

   C. $r \vee (p \wedge q)$

   D. $p \wedge q$

   E. None of the above.

4. Denote $A \triangle B$ to be $\bar{A} \cup B$. Which of the following sets is equal to $\big((A \triangle B) \triangle A\big) \triangle A$?

   A. A

   B. $A \cup B$

   C. $\bar{A} \cup B$

   D The universal set.

   E. None of the above.

5. Recall the definition of $A \oplus B$ on sets A and B:

$$A \oplus B = (A \backslash B) \cup (B \backslash A).$$

Suppose $A \cup B$ is the universal set, which of the following statements is equivalent to $x \in A \oplus B$?

A. $(x \in A \land x \in B) \lor (x \in B \land x \notin A)$
B. $(x \notin A \lor x \in B) \rightarrow (x \in B \land x \notin A)$
C. $(x \in A \land x \in B) \lor (x \notin B \land x \notin A)$
D $(x \in A \lor x \in B) \rightarrow (x \notin B \land x \notin A)$
E. None of the above.

6. Let $P(x)$ and $Q(x)$ be predicates. Suppose the following statement is true:

$$\forall x \in A \left(P(x) \land Q(x)\right).$$

Which of the following statements is not necessarily true?

A. $\left(\forall x \in A\, P(x)\right) \land \left(\forall x \in A\, Q(x)\right)$
B. $\left(\forall x \in A\, P(x)\right) \lor \left(\forall x \in A\, Q(x)\right)$
C. $\forall x \in A \left(P(x) \lor Q(x)\right)$
D $\forall x \in A \left(P(x) \rightarrow Q(x)\right)$
E. None of the above.

7. Which of the following statements is false?

A. $\exists x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ \exists z \in \mathbb{Z}\ (x - y < z)$
B. $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ \exists z \in \mathbb{Z}\ (x - y < z)$
C. $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}\ \exists z \in \mathbb{Z}\ (x - y < z)$
D $\forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ \exists z \in \mathbb{Z}\ (x - y < z)$
E. None of the above.

8. Let $A = \{a, b, c, d, e\}$ and R a relation on A such that

$$R = \{(a, b), (b, a), (b, c), (c, c), (c, d), (e, d)\}$$

Which of the following results in $R \cup S$ being the transitive closure of R?

A. $S = \{(a, c), (b, d)\}$
B. $S = \{(a, c), (b, d), (c, e)\}$
C. $S = \{(a, a), (b, b), (a, c), (b, d)\}$
D. $S = \{(a, a), (b, b), (d, d), (e, e), (a, c), (b, d)\}$
E. None of the above.

9. Let R^T be the transitive closure of the relation R in question 8 above. Which of the following are partial orders on A?

   (i) $R^T \cup \{(x,x) : x \in A\}$
   (ii) $(R^T \cup \{(x,x) : x \in A\}) \setminus \{(a,b)\}$
   (iii) $R^T \setminus \{(a,b)\}$
   (iv) $(R^T \cup \{(x,x) : x \in A\} \cup \{(c,e)\}) \setminus \{(a,b)\}$

   A. Only (i).
   B. Only (ii).
   C. Only (ii) and (iv).
   D. Only (ii), (iii) and (iv).
   E. None of the above.

10. Let $B = \{2,3,5,7,11,13,17,19,23\}$. Which of the following are equal to $B/{\sim}$ for some equivalence relation $\sim$ on B?

   (i) $\{\,\{2,3,5,7,11,13,17,19,23\}\,\}$
   (ii) $\{\,\{2\},\{3\},\{5\},\{7\},\{11\},\{13\},\{17\},\{19\},\{23\}\,\}$
   (iii) $\{\,\{2,3,5,7,11\},\{3,13,17,19,23\}\,\}$
   (iv) $\{\,\{2,3,5,11\},\{7,13,19,23\}\,\}$

   A. Only (i).
   B. Only (ii).
   C. Only (i) and (ii).
   D. Only (i), (ii) and (iii).
   E. Only (i), (ii) and (iv).

11. Let $\preccurlyeq$ be a partial order on $\mathbb{Z} \times \mathbb{Z}$ defined as follows: for all $(x_1, y_1), (x_2, y_2) \in \mathbb{Z} \times \mathbb{Z}$,

$$(x_1, y_1) \preccurlyeq (x_2, y_2) \quad \Leftrightarrow \quad (x_1 \le x_2) \wedge (y_1 \le y_2)$$

   where $\le$ is the usual less-than-or-equal symbol. Which of the following statements are true?

   (i) There is no largest element.
   (ii) $(1,2)$ and $(-4,2)$ are comparable.
   (iii) $(7,3)$ and $(2,9)$ are compatible.

   A. Only (i).
   B. Only (ii).
   C. Only (i) and (ii).
   D. Only (i) and (iii).
   E. All of (i), (ii) and (iii).

12. Given any relations R, S, T and U, which of the following statements are true?
    - (i) $(R^{-1})^{-1} = R$
    - (ii) $(R \circ S)^{-1} = R^{-1} \circ S^{-1}$
    - (iii) $(R \circ S) \circ T = T \circ (S \circ R)$
    - (iv) $(R \circ S) \circ (T \circ U) = R \circ (S \circ T) \circ U$

    A. Only (i).

    B. Only (ii) and (iii).

    C. Only (i) and (iv).

    D. Only (i), (ii) and (iv).

    E. All of (i), (ii), (iii) and (iv).

Part B (Total: 26 marks)

13. Let $\preccurlyeq$ be a partial order on a non-empty set A. A subset C of A is called a **chain** if and only if every pair of elements in C is comparable, that is, $\forall a, b \in C\ (a \preccurlyeq b \vee b \preccurlyeq a)$. A **maximal chain** is a chain M such that $t \notin M \Rightarrow M \cup \{t\}$ is not a chain. The length of a chain is one less than the number of elements in it. [5 marks]

    (a) Let $A = \{a, b, c, d\}$ and $(\mathcal{P}(A), \subseteq)$ be a poset on $\mathcal{P}(A)$, where $\mathcal{P}(A)$ denotes the power set of A. Write out two maximal chains in $(\mathcal{P}(A), \subseteq)$. [2]

    (b) Let $B = \{2,3,5,6,7,11,12,35,385\}$ and $(B, |)$ be a poset on B, where $|$ denotes the divides relation. Draw the Hasse diagram and write out two maximal chains of different lengths in $(B, |)$. [3]

14. The following **Resolution Rule** was introduced in assignment 1:

$$(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$$

Using the resolution rule, Theorem 2.1.1, rules of inference and proof by contradiction, prove that the following argument is valid. [6 marks]

$p \vee q$	(Premise 1: P1)
$q \rightarrow r$	(Premise 2: P2)
$p \wedge s \rightarrow t$	(Premise 3: P3)
$\sim r$	(Premise 4: P4)
$\sim q \rightarrow u \wedge s$	(Premise 5: P5)
$\therefore t$	(Conclusion)

Include numbering and justification on every step of your proof. The first 6 steps of the proof are given on the Answer Sheet.

Theorem 2.1.1 is given below for your convenience:

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$p \wedge q \wedge r \equiv (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$p \vee q \vee r \equiv (p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge \textbf{true} \equiv p$	$p \vee \textbf{false} \equiv p$
5	Negation laws	$p \vee \sim p \equiv \textbf{true}$	$p \wedge \sim p \equiv \textbf{false}$
6	Double negative law	$\sim(\sim p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee \textbf{true} \equiv \textbf{true}$	$p \wedge \textbf{false} \equiv \textbf{false}$
9	De Morgan's laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **true** and **false**	$\sim\textbf{true} \equiv \textbf{false}$	$\sim\textbf{false} \equiv \textbf{true}$

15. For each of the following statements, give an example for the statement to be true, and show that it is indeed true; if it is impossible for the statement to be true, write "false" and you don't need to prove it. [3 marks]

    (Each part is worth 1 mark. No partial credit will be given for each part.)

    (a) There exist non-empty finite sets A and B such that $|A \cup B| = |A| + |B|$.
    (b) There exist non-empty finite sets A and B such that $|A \cup B| \neq |A| + |B|$.
    (c) There exist finite sets A and B such that $A \times \mathcal{P}(B) = \mathcal{P}(A \times B)$.

16. For each part below, indicate whether the given relation is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, or transitive by writing a **T** (for true) or an **F** (for false) in the box for the said property on the Answer Sheet. For each part, only when you answer all properties correctly about that relation would you be awarded the mark. [5 marks]

    Note that a binary relation R on a set A is
    - **reflexive** iff $\forall x \in A\ (x, x) \in R$.
    - **irreflexive** iff $\forall x \in A\ (x, x) \notin R$.
    - **symmetric** iff $\forall x, y \in A\ \big((x, y) \in R \Rightarrow (y, x) \in R\big)$.
    - **antisymmetric** iff $\forall x, y \in A\ \big((x, y) \in R \wedge (y, x) \in R \Rightarrow x = y\big)$.
    - **asymmetric** iff $\forall x, y \in A\ \big((x, y) \in R \Rightarrow (y, x) \notin R\big)$.
    - **transitive** iff $\forall x, y, z \in A\ \big((x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R\big)$.

    Let $A = \{a, b, c, d\}$ and R_1, R_2, R_3, R_4, R_5 be binary relations on A such that:

    (a) $R_1 = A \times A$.
    (b) $R_2 = R_1 \setminus \{(x, x) : x \in A\}$.
    (c) $R_3 = \{(a, a),\ (a, b),\ (b, a),\ (b, b),\ (c, c),\ (c, d),\ (d, c)\}$.
    (d) $R_4 = \{(a, b),\ (a, c),\ (b, c),\ (d, d)\}$.
    (e) $R_5 = \emptyset$.

17. Please be reminded that you should include numbering and justification of important steps in your proof. Marks may also be deducted for proof that is unnecessarily long. [7 marks]

    (a) Let R be a binary relation on a set A. Given that R is symmetric and transitive, prove that R is an equivalence relation if for every $x \in A$, there exists $y \in A$ such that xRy. [3]

    (b) Let B be this set of prime numbers $\{2,3,5,7,11,13,17,19,23\}$. Define the relation S on B as follows:
    $$\forall x, y \in B\ (xSy \Leftrightarrow x|y)$$
    where $x|y$ denotes "x divides y".

    (i) Is S an equivalence relation? (Answer "yes" or "no".) If it is, what is $|B/S|$? [2]

    (ii) Is S a partial order? (Answer "yes" or "no".) If it is, write out all the maximal elements and all the minimal elements. [2]

=== END OF PAPER ===