NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING

MID-TERM TEST
AY2023/24 Semester 1

CS1231S — DISCRETE STRUCTURES

11 October 2023 Time Allowed: 1 hour 30 minutes

INSTRUCTIONS

1. This assessment paper contains **NINETEEN (19)** questions in **TWO (2)** parts and comprises **SEVEN (7)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions.
5. Write your answers only on the **ANSWER SHEET**. You may write in pen or pencil, except for shading your Student Number and MCQ answers, which you must use pencil (2B or above).
6. You are to write within the space provided. No extra pages should be submitted as it will jam the scanning machine.
7. The maximum mark of this assessment is 50.
8. Do not start writing or flip over this page until you are told to do so.

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions (Total: 30 marks)

Each multiple choice question (MCQ) is worth two marks and has exactly **one** correct answer. (Except for question 1. However, you are to indicate only one answer for question 1.)

1. Which of the following was mentioned by Aaron Tan in his lectures this semester? (Note: There could be more than one correct answer, but you are to select ONLY ONE answer. If you select more than one answer, you will NOT get any mark, even if all the answers you select are correct.)

   A. Mr Alton was looking at Ms Betty and Ms Betty was looking at Mr Carl.

   B. There are 10 types of people: those who understand binary numbers and those who don't.

   C. Mother told son, "if you behave, you will get ice-cream".

   D. What is the volume of a thick crust pizza with height a and radius z?

   E. None of the above was told in lectures this semester, so this is a trick question.

2. Let S be the set of students and $x \in S$. We define the following predicates:

   $SoC(x)$: x is a computing student;

   $CS1231(x)$: x has taken CS1231S.

   Given that the statement $\forall x \in S\ \big(SoC(x) \wedge CS1231(x)\big)$ is true, which of the following statements are true?

   (i) All computing students have taken CS1231S.

   (ii) Every student is a computing student.

   (iii) There must be at least one student in the set S.

   (iv) Some non-Computing student has taken CS1231S.

   A. Only (i).

   B. Only (i) and (iii).

   C. Only (ii) and (iv).

   D. Only (i), (ii) and (iv).

   E. None of the options (A), (B), (C), (D) is correct.

3. Which of the following statements are true?

   (i) $\sim(p \to q) \equiv \big(p \vee (p \to \sim q)\big)$

   (ii) $\big(p \to (q \vee r)\big) \leftrightarrow \big((p \wedge \sim q) \to r\big)$

   (iii) $\big((p \leftrightarrow q) \wedge (q \to r) \wedge (r \vee \sim s) \wedge (\sim s \to q)\big) \to s$

   A. Only (ii).

   B. Only (i) and (ii).

   C. Only (i) and (iii).

   D. Only (ii) and (iii).

   E. All of (i), (ii) and (iii).

4. The island of Wantuutrewan is inhabited by exactly two types of natives: **knights** who always tell the truth and **knaves** who always lie. Every native is a knight or a knave, but not both.

You meet three islanders and here is their conversation:

Aiken says: Dueet tells the truth.

Dueet says: Noasu never tells the truth.

Noasu says: Aiken is not my type.

Noasu says: Aiken lies.

Which of the following is true?

A. There are no knights amongst the three islanders.
B. Aiken is a knave.
C. Dueet is a knight.
D. Noasu is a knave.
E. None of the above.

5. Assume the domain is $\mathbb{R}$. Which of the following statements are true?

(i) $\forall x\, \exists y\, (xy = 1)$
(ii) $\exists x\, \forall y\, (x^2 \geq y^2)$
(iii) $\exists x\, \forall y\, \exists z\, (xy^2 = yz)$

A. Only (ii).
B. Only (iii).
C. Only (i) and (ii).
D. Only (i) and (iii).
E. None of the options (A), (B), (C), (D) is correct.

6. Which of the following sets are equal to $\emptyset \cup \mathcal{P}(\emptyset)$?

(i) $\emptyset \times \{\emptyset\}$
(ii) $\mathcal{P}\big(\emptyset \cup \mathcal{P}(\emptyset)\big)$
(iii) $\mathcal{P}(\emptyset) \setminus \emptyset$
(iv) $(\emptyset \cap \{\emptyset\}) \cup (\emptyset \cup \{\emptyset\})$

A. Only (ii).
B. Only (iv).
C. Only (i) and (iii).
D. Only (iii) and (iv).
E. None of the options (A), (B), (C), (D) is correct.

7. For which of the following expressions you cannot find two non-empty sets A and B satisfying it? [Note: The $\oplus$ operator is defined in tutorial 3 as follows: $A \oplus B = (A \setminus B) \cup (B \setminus A)$.]

    (i) $A \setminus B = A \cap B$
    (ii) $A \setminus B = A \cup B$
    (iii) $A \cup B = A \cap B$
    (iv) $A \oplus B = A \cup B$

    A. Only (i).
    B. Only (i) and (ii).
    C. Only (ii) and (iv).
    D. Only (i), (ii) and (iv).
    E. None of the options (A), (B), (C), (D) is correct.

8. Which of the following statements are true for any two sets A and B?

    (i) $(A \in B) \wedge (B \subseteq C) \rightarrow A \in C$
    (ii) $(A \in B) \wedge (B \subseteq C) \rightarrow A \subseteq C$
    (iii) $(A \subseteq B) \wedge (B \in C) \rightarrow A \in C$
    (iv) $(A \subseteq B) \wedge (B \in C) \rightarrow A \subseteq C$

    A. Only (i).
    B. Only (i) and (ii).
    C. Only (i), (ii) and (iv).
    D. All of (i), (ii), (iii) and (iv).
    E. None of the options (A), (B), (C), (D) is correct.

9. Let R and S be two relations on a non-empty set A. R is a reflexive relation and S is a symmetric relation. Which of the following statements are true?

    (i) $R \cup S$ is a reflexive relation.
    (ii) $R \circ S$ is a symmetric relation.
    (iii) $R \cup S$ is a symmetric relation.
    (iv) $(R \circ S)^{-1}$ is a transitive relation.

    A. Only (i).
    B. Only (i) and (iv).
    C. Only (ii) and (iii).
    D. Only (i), (ii) and (iv).
    E. None of the options (A), (B), (C), (D) is correct.

10. Consider the following statements about relations on any non-empty set:

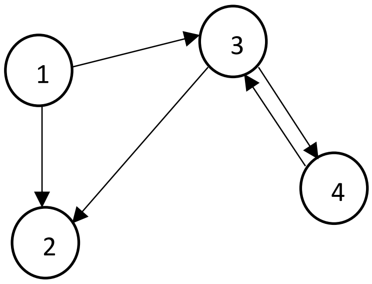
    (i) Every asymmetric relation is antisymmetric.
    (ii) If a relation is not reflexive then it is not asymmetric.
    (iii) If a relation is not reflexive then it is asymmetric.
    (iv) There are no relations that are both an equivalence relation and a partial order.

    Which of the above statements are true?

    Note: Asymmetry is defined in Tutorial 5 as follows:
    A binary relation R on a set A is asymmetric iff $\forall x, y \in A\ (xRy \Rightarrow y \not{R} x)$.

    A. Only (i).
    B. Only (ii).
    C. Only (i) and (iv).
    D. Only (ii) and (iii).
    E. None of the options (A), (B), (C), (D) is correct.

11. The directed graph for a relation R on $\{1,2,3,4\}$ is given below. What is $|R \circ R \circ R \circ R|$?



    A. 3.
    B. 5.
    C. 7.
    D. 9.
    E. None of the above.

12. Which of the following are always true for all equivalence relations R?

    (i) $R^{-1} \circ R = R \circ R^{-1}$
    (ii) $R \circ R^{-1} = R$
    (iii) $R \circ R \subseteq R$
    (iv) $R \subseteq R \circ R$

    A. Only (i) and (ii).
    B. Only (iii) and (iv).
    C. Only (i), (iii) and (iv).
    D. All of (i), (ii), (iii) and (iv).
    E. None of the options (A), (B), (C), (D) is correct.

13. Which of the following statements are true about non-empty partial order relations?

    (i) It is not possible to have an element that is both smallest and largest.

    (ii) It is possible to have an element that is both maximal and minimal but neither largest nor smallest.

    (iii) Different maximal elements are comparable to each other.

    (iv) It is possible to not have any maximal or minimal elements.

    A. Only (ii).

    B. Only (i) and (ii).

    C. Only (ii) and (iv).

    D. Only (ii), (iii) and (iv).

    E. None of the options (A), (B), (C), (D) is correct.

14. Let $A = \{1,2,3\}$ and define the following relations on A:

    (i) $P = \{(1,1), (1,2), (2,2)\}$

    (ii) $Q = \{(1,1), (1,2), (1,3), (2,2), (3,2), (3,3)\}$

    (iii) $R = \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,1)\}$

    (iv) R^t, the transitive closure of R.

    Which of the above relations are partial orders?

    A. Only P.

    B. Only Q.

    C. Only Q and R^t.

    D. Only P and Q.

    E. None of the options (A), (B), (C), (D) is correct.

15. Let $A = \{11,12,13,14,15,16\}$. For each $x \in A$, define $F_x = \{k \in \mathbb{Z}^+ : k|x\}$, where $|$ is the "divides" relation. Define also a partial order $\preccurlyeq$ on A by setting for all $x, z \in A$:

    $$x \preccurlyeq z \iff (F_x = F_z) \vee (|F_x| < |F_z|).$$

    (This is the same partial order in Q18.)

    Which of the following are linearizations $\preccurlyeq^*$ of $\preccurlyeq$?

    (i) $11 \preccurlyeq^* 12 \preccurlyeq^* 13 \preccurlyeq^* 14 \preccurlyeq^* 15 \preccurlyeq^* 16$

    (ii) $13 \preccurlyeq^* 11 \preccurlyeq^* 15 \preccurlyeq^* 14 \preccurlyeq^* 16 \preccurlyeq^* 12$

    (iii) $11 \preccurlyeq^* 13 \preccurlyeq^* 14 \preccurlyeq^* 15 \preccurlyeq^* 16 \preccurlyeq^* 12$

    (iv) $11 \preccurlyeq^* 13 \preccurlyeq^* 12 \preccurlyeq^* 16 \preccurlyeq^* 14 \preccurlyeq^* 15$

    A. Only (i).

    B. Only (ii) and (iii).

    C. Only (ii) and (iv).

    D. Only (iii) and (iv).

    E. None of the options (A), (B), (C), (D) is correct.

Part B (Total: 20 marks)

16. Answer the following parts. Note that $|x|$ denotes the absolute value of x. [6 marks]

    (a) Let $A = \{1,2,3\}$ and $B = \{4,5,6\}$. Assuming that C is a non-empty set containing positive integers, what is the smallest possible element in C such that the following statement is true?

    $$\forall x \in A\ \forall y \in B\ \forall z \in C\ (|x - y| \leq |y - z|).$$ (2 marks)

    (b) Let $A = \{10,20,30,40,50\}$ and $C = \{3,5,7,9\}$. Write out all the values of y that would make the following statement true.

    $$\exists y \in \mathbb{Z}^+\ \forall x \in A\ \forall z \in C\ \big((x \leq y^2) \wedge (y \leq z^2)\big).$$ (2 marks)

    (c) The following statement is true. For each value of x, what is the value of y that makes the statement true?

    $$\forall x \in \mathbb{Z}\ \exists y \in \mathbb{R}\backslash\{0\}\ (xy < 1).$$ (2 marks)

17. Let $A_1 = \{\emptyset\}, A_2 = \{a\}$ and $A_3 = \{a, \{a\}\}$. Write out the following sets. [4 marks]

    (a) $(A_1 \cap A_2) \setminus A_3$

    (b) $(A_1 \cup A_2) \setminus A_3$

    (c) $(A_1 \times A_2) \cup A_3$

    (d) $(A_1 \cup A_2) \times A_3$

18. Let $A = \{11,12,13,14,15,16\}$. For each $x \in A$, define $F_x = \{k \in \mathbb{Z}^+ : k|x\}$, where $|$ is the "divides" relation. Define also a partial order $\preccurlyeq$ on A by setting for all $x, z \in A$:

    $$x \preccurlyeq z \iff (F_x = F_z) \vee (|F_x| < |F_z|).$$

    (This is the same partial order in Q15.)

    What are the (a) minimal, (b) smallest, (c) maximal, and (d) largest elements of A with respect to $\preccurlyeq$? If the answer is none, write "none". Do not leave it blank or it will be treated as a non-answer. [4 marks]

19. Define a relation S on $\mathbb{Q}$ by setting for all $a, b \in \mathbb{Q}$:

    $$aSb \iff \exists k \in \mathbb{Z}\left(a - b = \tfrac{1}{2}k\right).$$ [6 marks]

    (a) S is reflexive and symmetric. Prove that S is transitive. (3 marks)

    (b) Write out two distinct equivalence classes of S. (1 mark)

    (c) Write out $\mathbb{Q}/S$ in set-builder notation or set-replacement notation. You are to ensure that your set does <u>not</u> contain duplicate elements. (2 marks)

=== END OF PAPER ===