CS1231S Discrete Structures

Midterm Test — Answer Sheets

AY2023/24 Semester 2

Time allowed: 1 hour 30 minutes

INSTRUCTIONS

ANSWERS

1. Write your **Student Number** and **Tutorial Group Number** (eg: T02) on the right. Do not write your name.
2. Check that you have written your Student Number correctly. No mark will be awarded for this test if we are unable to identify you with your Student Number.
3. These answer sheets comprise **FOUR (4) pages**.
4. All questions must be answered in the space provided; no extra sheets will be accepted as answers.
5. You must submit only these **ANSWER SHEETS** and no other documents.
6. An excerpt of the question may be provided to aid you in answering in the correct box. It is not the exact question. You should still refer to the original question in the question paper.
7. You may write your answers using pencil (at least 2B) or pen as long as it is legible (no red ink, please).
8. The maximum mark for this paper is 50.
9. **Marks may be deducted** for (i) illegible handwriting, and/or (ii) excessively long answer.
10. Each multiple choice question is intended to have only one answer. Please shade the appropriate bubble.

My Student Number:

A								

My Tutorial Grp Number: T

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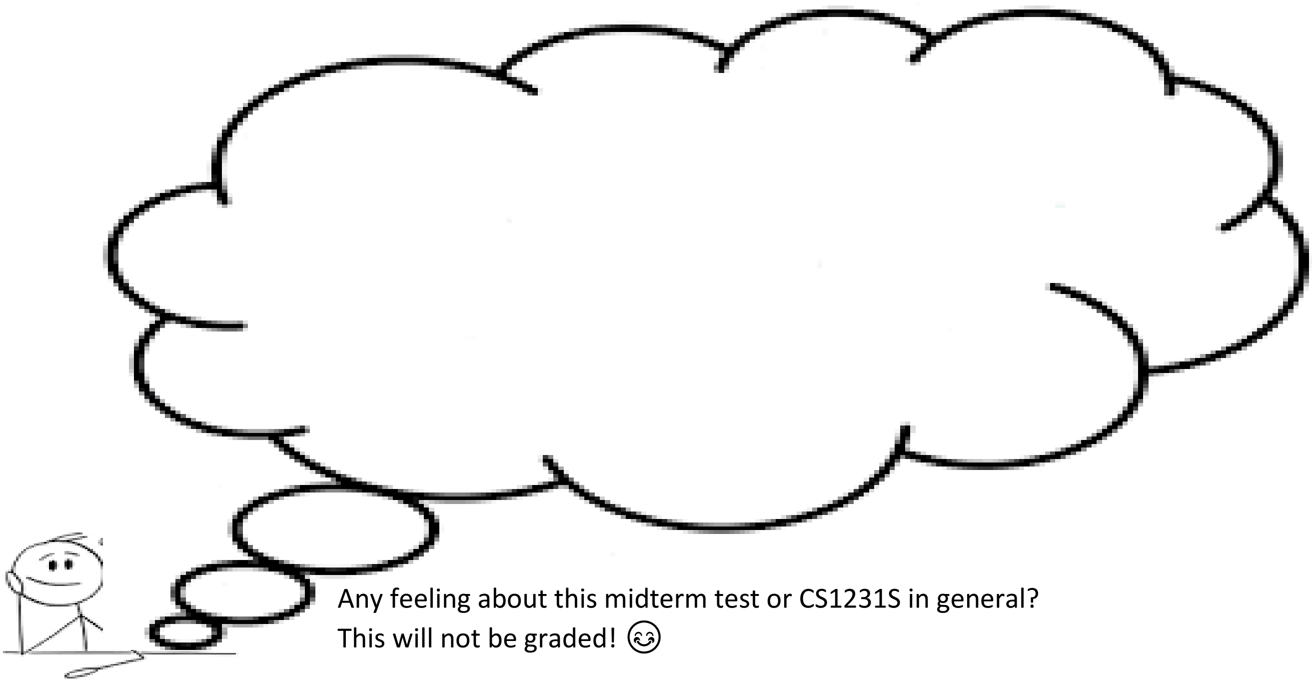
Question	**Marks**	**Remarks**
Q1-15	/ 30	
Q16	/ 2	
Q17	/ 6	
Q18	/ 4	
Q19	/ 8	
Total	**/ 50**	



Any feeling about this midterm test or CS1231S in general?
This will not be graded! 😋

Explanation of MCQs

3. (i) true; (ii) q; (iii) $\sim q$; (iv) q.

5. (i) is false. If $x > y$, there is no m such that $(x < m) \land (m < y)$.
   (ii) is true. For any $x \in \mathbb{R}\backslash\{0\}$, let $y = -\frac{1}{x}$, then $xy = -1 < 0$.
   (iii) is false. Counterexample: for any x, y, let $z = xy$.

6. (i) is false. For $x = 1$, there are two values of y (-1 and 1) such that $x = y^2$.
   (ii) is true. Let $x = 0$.
   (iii) is false. Counterexample: Let $x = 1, y = 0, z = 2$.

7. (i) is false. Counterexample: Let $A = \{a\}, B = \{b\}$. Then $A \cup B = \{a, b\}, \mathcal{P}(A) = \{\emptyset, \{a\}\}, \mathcal{P}(B) = \{\emptyset, \{b\}\}, \mathcal{P}(A \cup B) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \neq \{\emptyset, \{a\}, \{b\}\} = \mathcal{P}(A) \cup \mathcal{P}(B)$.
   (ii) is true. Proof omitted.
   (iii) is false. If $A \subseteq \emptyset$, then $A = \emptyset$ (since $\emptyset \subseteq A$). Hence $\mathcal{P}(A) = \{\emptyset\}$ and $\mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}\}$.

8. The number of cases to check is $6^3 = 216$.

9. Let a, b, c, d, e be the elements of A from bottom to top in the linearization. As linearization is a total order, a is related to 5 elements, b to 4, c to 3, d to 2 and e to 1. Hence, 5+4+3+2+1 = 15.

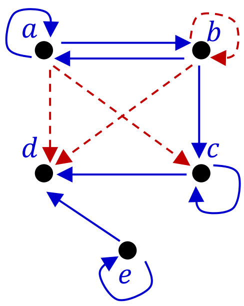
10. $R \circ R = \{(1,1),\ (1,4),\ (2,2),\ (2,3),\ (3,2),\ (4,1)\}$.
    $R \circ R \circ R \circ R = (R \circ R) \circ (R \circ R) = \{(1,1),\ (1,4),\ (2,2),\ (2,3), (3,2),\ (3,3),\ (4,1),\ (4,4)\}$.

11. Maximal elements are $\{12,18,15,10\}$. Minimal elements are $\{2,3\}$.

12. There will be <u>no</u> total order that is a linearization of $\preccurlyeq_1$ and $\preccurlyeq_2$ as 4 and 5 are in contradictory order.
    A total order that is a linearization of $\preccurlyeq_1$ and $\preccurlyeq_3$ is shown in the left diagram.
    A total order that is a linearization of $\preccurlyeq_2$ and $\preccurlyeq_3$ is shown in the right diagram.

1	1
|	|
5	4
|	|
3	5
|	|
4	3
|	|
2	2

13. The correct answer is $S = \{(b, b), (b, d), (a, c), (a, d)\}$.



14. (A) is false. $\mathbb{Z}^-$ is a non-empty subset of $\mathbb{Z}$ that has no smallest element.
    (B) is false. The divisibility relation is not a total order.
    (C) is false. The $=$ relation is not a total order.
    (D) is false. Take $\mathbb{Q}^+$ (which is a subset of $\mathbb{Q}^+$). There is no smallest element in $\mathbb{Q}^+$.

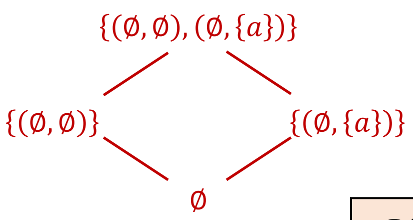
Part A: Multiple Choice Questions (Total: 30 marks)

Please shade using **pencil** only (2B or above) ONE bubble for each question.

	(A)	(B)	(C)	(D)	(E)
1.	●	○	○	○	○
2.	○	○	●	○	○
3.	○	●	○	○	○
4.	○	○	○	●	○
5.	○	○	○	○	●
6.	●	○	○	○	○
7.	○	●	○	○	○
8.	○	○	●	○	○
9.	○	●	○	○	○
10.	○	○	○	○	●
11.	○	○	●	○	○
12.	○	○	●	○	○
13.	○	○	○	○	●
14.	○	○	○	○	●
15.	○	○	○	●	○
	(A)	(B)	(C)	(D)	(E)

Part B (Total: 20 marks)

16. [2 marks]



$$\mathcal{P}(\emptyset)\times\mathcal{P}(\{a\}) = \{\emptyset\}\times\{\emptyset,\{a\}\} = \{(\emptyset,\emptyset),(\emptyset,\{a\})\}$$

17. [6 marks]

(a) [1]

$$\bigcup_{i=4}^{14} A_i = \{3, 6, 9, 12\}$$

(b) [1]

$$\bigcap_{i=4}^{14} A_i = \{3\}$$

(c) [2]

Let $k = 3q + r$ such that $0 \le r < 3$.
$|A_k| = q$.

(d) [2]

Let $k = 3q + r$ such that $0 \le r < 3$.
Number of equivalence classes is $q + 1$.

Note that $A_1 = A_2 = \emptyset$; $A_3 = A_4 = A_5 = \{3\}$; $A_6 = A_7 = A_8 = \{3,6\}$; $A_9 = A_{10} = A_{11} = \{3,6,9\}$; etc.

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18. [4 marks]

(a) [1]

R is reflexive and not irreflexive.

Let $A = \{a\}$ and $R = \{(a, a)\}$.

Many other possible examples. You should give the simplest examples.

Note that simply writing $R = \{(a, a)\}$ for (a), $R = \emptyset$ for (b), and $R = \{(a, a)\}$ for (c) are <u>incorrect</u>, because what A contains matters.

If $A = \{a\}$, then the relation $R = \{(a, a)\}$ is reflexive, but if $A = \{a, b\}$, then the relation $R = \{(a, a)\}$ is <u>not</u> reflexive.

(b) [1]

R is irreflexive and not reflexive.

Let $A = \{a\}$ and $R = \emptyset$.

Let $A = \{a, b\}$ and $R = \{(a, b)\}$.

(c) [1]

R is not reflexive and not irreflexive.

Let $A = \{a, b\}$ and $R = \{(a, a)\}$.

(d) [1]

R is reflexive and irreflexive.

Let $A = \emptyset$ (and hence $R = \emptyset$.)

Reflexivity and irreflexivity are both vacuously true. Note: Simply writing $R = \emptyset$ is not correct, as A can be non-empty, in which case R is irreflexive and not reflexive.

19. [8 marks] Write (a), (b), (c), (d) to indicate the part. Draw a line to separate the answers for different parts.

(a) [1 mark] If R is reflexive and symmetric, then R is transitive. → **FALSE**
Counterexample: Let $A = \{x, y, z\}$ and
$R = \{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y)\}$.
R is reflexive and symmetric, but it is not transitive as $(x, z) \notin R$.

(b) [1 mark] If R is symmetric and transitive, then R is reflexive. → **FALSE**
Counterexample: Let $A = \{x, y, z\}$ and $R = \{(x, x), (y, y), (x, y), (y, x)\}$.
R is symmetric and transitive, but it is not reflexive as $(z, z) \notin R$.
A simpler counterexample: Let $A = \{x, y\}$ and $R = \{(x, x)\}$.
An even simpler counterexample: Let $A = \{x\}$ and $R = \emptyset$.

(c) [3 marks] If R is reflexive and asymmetric, then R is not transitive. → **TRUE**
Note that a relation R on a non-empty set A cannot be both reflexive and asymmetric, as it is contradictory. Proof:

1. Let R be a reflexive and asymmetric relation on a non-empty set A.
2. Then xRx for some $x \in A$ as R is reflexive and A is non-empty.
3. Then $x \not R x$ as R is asymmetric, which contradicts 2.
4. Since R is reflexive and asymmetric is false, the statement is vacuously true.

(d) [3 marks] If R is reflexive and asymmetric, then R is not transitive. → **TRUE**
Note that only an empty relation $R = \emptyset$ can be both symmetric and asymmetric.
Proof:

1. Let R be a symmetric and asymmetric relation on a non-empty set A.
2. Suppose $R \neq \emptyset$, i.e. some $x, y \in A$ such that $(x, y) \in R$.
3. Then $(y, x) \in R$ (by symmetry) and $(y, x) \notin R$ (by asymmetry), which are contradictory.
4. Hence, $R = \emptyset$.
5. Therefore, R is transitive (vacuously true).

=== END OF PAPER ===