NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING

MID-TERM TEST
AY2023/24 Semester 2

CS1231S — DISCRETE STRUCTURES

13 March 2024 Time Allowed: 1 hour 30 minutes

INSTRUCTIONS

1. This assessment paper contains **NINETEEN (19)** questions in **TWO (2)** parts and comprises **SEVEN (7)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions.
5. Write your answers only on the **ANSWER SHEETS** provided. You are to write within the space provided. No extra pages should be submitted.
6. Write legibly, or marks may be deducted. You may write in pen or pencil. Make sure your ink is dark and your pencil is 2B or above.
7. The maximum mark of this assessment is 50.
8. Do not start writing or flip over this page until you are told to do so.

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions (Total: 30 marks)

Each multiple choice question (MCQ) is worth <u>two marks</u> and has exactly **one** correct answer.

1. Which lecture theatre are CS1231S lectures conducted in this semester?

    A. LT15
    B. LT16
    C. LT19
    D. UTown Auditorium 2
    E. This is a trick question. The lectures are conducted online.

2. The island of Wantuutrewan is inhabited by exactly two types of natives: **knights** who always tell the truth and **knaves** who always lie. Every native is a knight or a knave, but not both.

    You meet two islanders and here is what they say:

    Aiken says: Dueet and I are the same type of people.
    Dueet says: Aiken and I are different types of people.

    Which of the following is true?

    A. Both Aiken and Dueet are knights.
    B. Aiken is a knight and Dueet is a knave.
    C. Aiken is a knave and Dueet is a knight.
    D. Both Aiken and Dueet are knaves.
    E. None of the above (the puzzle cannot be solved).

3. Given the following where p, q, r, s, t are statement variables:

    (i) $(\sim r \rightarrow \sim r) \vee (\sim q \rightarrow q) \vee (t \rightarrow \sim t)$
    (ii) $(p \wedge q \wedge s) \vee (\sim p \wedge q \wedge \sim r \wedge t) \vee q \vee (q \wedge r \wedge s \wedge \sim t)$
    (iii) $\big((t \wedge s) \rightarrow (t \vee s)\big) \wedge (q \rightarrow \sim q)$
    (iv) $(\sim q \rightarrow q) \wedge (r \rightarrow r)$

    Which of the above are logically equivalent to one another?

    A. Only (i) and (iii).
    B. Only (ii) and (iv).
    C. Only (i), (ii) and (iv).
    D. Only (ii), (iii) and (iv).
    E. None of the options (A), (B), (C), (D) is correct.

4. Let p and q be statement variables. Which of the following is the negation of $p \rightarrow q$?

   A. $\sim p \rightarrow \sim q$
   B. $\sim p \vee q$
   C. $p \rightarrow \sim q$
   D. $\sim q \wedge p$
   E. None of the options (A), (B), (C), (D) is correct.

5. Consider the following statements:

   (i) $\forall x, y \in \mathbb{Q}\ \big(x \neq y \rightarrow \exists m \in \mathbb{Q}\ ((x < m) \wedge (m < y))\big)$.
   (ii) $\forall x \in \mathbb{R}\backslash\{0\}\ \exists y \in \mathbb{R}\ (xy < 0)$.
   (iii) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ \forall z \in \mathbb{R}\ (xy < z)$.

   Which of the above statements are true?

   A. Only (i).
   B. Only (i) and (ii).
   C. Only (ii) and (iii).
   D. All of (i), (ii) and (iii).
   E. None of the options (A), (B), (C), (D) is correct.

6. Given the sets $A = \{0,1,4\}, B = \{-2, -1,0,1,2\}$ and $C = \{1,2,4\}$, consider the following statements:

   (i) $\forall x \in A\ \exists!\, y \in B\ (x = y^2)$.
   (ii) $\exists x \in A\ \forall y \in B\ \forall z \in C (xy < z)$.
   (iii) $\forall x \in A\ \forall y \in B\ \forall z \in C\ (x < z \rightarrow x \leq y^2)$.

   Which of the above statements are true?

   A. Only (ii).
   B. Only (iii).
   C. Only (i) and (ii).
   D. Only (ii) and (iii).
   E. None of the options (A), (B), (C), (D) is correct.

7. Let A and B be sets. $\mathcal{P}(X)$ denotes the power set of set X. Consider the following statements:

   (i) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

   (ii) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

   (iii) $A \subseteq \emptyset \rightarrow \left|\mathcal{P}\big(\mathcal{P}(A)\big)\right| = 1$.

   Which of the above statements are true?

   A. Only (i).
   B. Only (ii).
   C. Only (ii) and (iii).
   D. All of (i), (ii) and (iii).
   E. None of the options (A), (B), (C), (D) is correct.

8. Given a set A with 6 elements and a relation R on A, how many cases in total do we need to check for an exhaustive check on the transitivity of R?

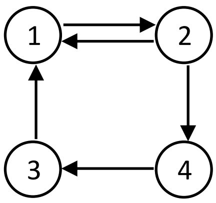
   A. 18
   B. 36
   C. 216
   D. 729
   E. None of the options (A), (B), (C), (D) is correct.

9. Let $\preccurlyeq$ be a partial order on a set A where $|A| = 5$, and let $\preccurlyeq^*$ be a linearization on $\preccurlyeq$. What is the cardinality of $\preccurlyeq^*$?

   A. 10
   B. 15
   C. 25
   D. Cannot be determined as the cardinality of $\preccurlyeq$ is not given.
   E. None of the options (A), (B), (C), (D) is correct.

10. The directed graph for a relation R on $\{1,2,3,4\}$ is given below. What is $|R \circ R \circ R \circ R|$?

   A. 3.
   B. 5.
   C. 7.
   D. 9.
   E. None of the above.

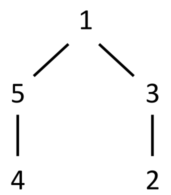


11. Given the poset $(\{2,3,4,6,9,10,12,15,18\}, |)$ where | is the divisibility symbol, which of the following statements is true?

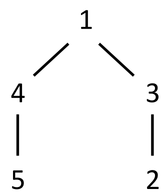
    A. There are 3 maximal elements and 2 minimal elements.
    B. There are 4 maximal elements and 3 minimal elements.
    C. There are 4 maximal elements and 2 minimal elements.
    D. There are 3 maximal elements and 3 minimal elements.
    E. None of the options (A), (B), (C), (D) is correct.

12. The following are Hasse diagrams of the partial orders $\preccurlyeq_1, \preccurlyeq_2$ and $\preccurlyeq_3$ on $\{1,2,3,4,5\}$.

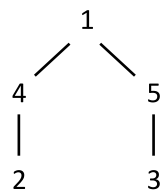
    $\preccurlyeq_1$:



    $\preccurlyeq_2$:



    $\preccurlyeq_3$:



    Consider the following statements:

    (i) There is a total order on $\{1,2,3,4,5\}$ that is a linearization of only $\preccurlyeq_1$ and $\preccurlyeq_2$.
    (ii) There is a total order on $\{1,2,3,4,5\}$ that is a linearization of only $\preccurlyeq_1$ and $\preccurlyeq_3$.
    (iii) There is a total order on $\{1,2,3,4,5\}$ that is a linearization of only $\preccurlyeq_2$ and $\preccurlyeq_3$.
    (iv) There is a total order on $\{1,2,3,4,5\}$ that is a linearization of all of $\preccurlyeq_1$, $\preccurlyeq_2$ and $\preccurlyeq_3$.

    Which of the above statements are true?

    A. Only (i).
    B. Only (ii).
    C. Only (ii) and (iii).
    D. Only (iii) and (iv).
    E. None of the options (A), (B), (C), (D) is correct.

13. A relation R on the set $A = \{a, b, c, d, e\}$ is given as follows:

$$R = \{(a,a),\ (a,b),\ (b,a),\ (b,c),\ (c,c),\ (c,d),\ (e,d),\ (e,e)\}.$$

    Let S be such that $R \cup S$ is the transitive closure of R. Which of the following is the correct S?

    A. $S = \{(a,c),\ (b,d),\ (a,d)\}$.
    B. $S = \{(b,b),\ (a,c),\ (b,d),\ (a,d),\ (d,d)\}$.
    C. $S = \{(b,b),\ (b,d),\ (d,d),\ (c,b),\ (d,c),\ (d,e)\}$.
    D. $S = \{(b,b),\ (d,d),\ (b,d),\ (d,b),\ (c,b),\ (d,c),\ (d,e),\ (a,c),\ (c,a)\}$.
    E. None of the options (A), (B), (C), (D) is correct.

14. Which of the following is a well-ordered set?
    - A. The set $\mathbb{Z}$ with the relation $\leq$ (less than or equal to) defined on it.
    - B. The set $\mathbb{Z}^+$ with the relation $|$ (divisibility relation) defined on it.
    - C. The set $\mathbb{N}$ with the relation $=$ (equal to) defined on it.
    - D. The set $\mathbb{Q}^+$ with the relation $\leq$ (less than or equal to) defined on it.
    - E. None of the options (A), (B), (C), (D) is correct.

15. In class, we gave the example of a mother telling her son this rule: "If you behave, then you get ice-cream." Consider the following statements:
    - (i) "You behave" is a necessary and sufficient condition for "you get ice-cream".
    - (ii) "You get ice-cream" is a necessary condition for "you behave".
    - (iii) If the son does not behave and gets ice-cream, the rule is still not broken.

    Which of the above statements are true?
    - A. Only (i).
    - B. Only (ii).
    - C. Only (i) and (ii).
    - D. Only (ii) and (iii).
    - E. None of the options (A), (B), (C), (D) is correct.

Part B (Total: 20 marks)

Write your answers within the space on the Answer Sheets. Do not attach additional sheets of paper as they will be disregarded.

16. Given the poset $\left(\mathcal{P}\left(\mathcal{P}(\emptyset) \times \mathcal{P}(\{a\})\right), \subseteq\right)$, draw the Hasse diagram for this poset.
    (Note: $\mathcal{P}(X)$ is the power set of the set X.) [2 marks]

17. [6 marks] Let $i \in \mathbb{Z}^+$. The set A_i is defined as follows:

$$A_i = \{n \in \mathbb{Z}^+ : (n \leq i) \wedge (n = 3k \text{ for some } k \in \mathbb{Z})\}$$

   (a) What is $\bigcup_{i=4}^{14} A_i$? Write your answer in set-roster notation. [1 mark]

   (b) What is $\bigcap_{i=4}^{14} A_i$? Write your answer in set-roster notation. [1 mark]

   Recall Theorem 4.4.1 (The Quotient-Remainder Theorem):

   > "Given any integer n and positive integer d, there exist unique integers q and r such that $n = dq + r$ and $0 \leq r < d$."

   (c) For a given positive integer k, what is the cardinality of A_k? [2 marks]

   (d) Define a relation $\sim$ on $\{A_1, A_2, A_3, \cdots, A_k\}$, where $k \in \mathbb{Z}^+$, as follows:

$$A_i \sim A_j \text{ if, and only if, } |A_i| = |A_j|.$$

   $\sim$ is found to be an equivalence relation on $\{A_1, A_2, A_3, \cdots, A_k\}$. How many equivalence classes are there on $\{A_1, A_2, A_3, \cdots, A_k\}$ with respect to $\sim$? [2 marks]

18. Let R be a binary relation on a set A. We define irreflexivity of a relation as follows:

$$R \text{ is irreflexive if, and only if, } \forall x \in A\ (x, x) \notin R.$$

Give an example for each of the following. Do not draw diagrams, as they will be disregarded. You do not need to explain your example. If it is impossible to give an example (because the statement is false), write "Impossible" and give an explanation, instead of leaving it blank. [4 marks]

(a) R is reflexive and not irreflexive.

(b) R is irreflexive and not reflexive.

(c) R is not reflexive and not irreflexive.

(d) R is reflexive and irreflexive.

19. Let R be a binary relation on a non-empty set A. State whether each of the following statements is true or false. Write **True** or **False** in full in the answer sheets, instead of T or F. If it is true, prove it; if it is false, give a counterexample. Note: Merely stating True or False without a proof or counterexample will be disregarded even if the truth value is correct. Presenting a proof or counterexample without first stating True or False will also be disregarded. Do not draw diagrams for your proof or counterexample. [8 marks]

(a) If R is reflexive and symmetric, then R is transitive.

(b) If R is symmetric and transitive, then R is reflexive.

(c) If R is reflexive and asymmetric, then R is not transitive.

(d) If R is symmetric and asymmetric, then R is transitive.

=== END OF PAPER ==