CS1231S: Discrete Structures

Tutorial #3: Sets

(Week 5: 9 – 13 September 2024)

Answers

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2. Tutorial Questions

Note that the sets here are finite sets, unless otherwise stated.

1. Let $\mathcal{P}(A)$ denotes the power set of A. Find the following:

   a. $\mathcal{P}(\{a, b, c\})$; ***Answer:*** $\mathcal{P}(\{a, b, c\}) = \{\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a, b\},\ \{a, c\},\ \{b, c\},\ \{a.b, c\}\}$.

   b. $\mathcal{P}\left(\mathcal{P}(\mathcal{P}(\emptyset))\right)$. ***Answer:*** $\mathcal{P}\left(\mathcal{P}(\mathcal{P}(\emptyset))\right) = \left\{\emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\emptyset, \{\emptyset\}\}\right\}$.

2. Let the universal set be $\mathbb{R}$, $A = \{x \in \mathbb{R} : -2 \le x \le 1\}$ and $B = \{x \in \mathbb{R} : -1 < x < 3\}$.

   Note that *A* and *B* may be written as [-2,1] and (-1,3) respectively using real numbers interval notation (Lecture #5 slide 27). Find the following and write your answers in set-builder notation, and if appropriate, also in interval notation.

   a. $A \cup B$; ***Answer:*** $\{x \in \mathbb{R} : -2 \le x < 3\}$ or $[-2,3)$

   b. $A \cap B$; ***Answer:*** $\{x \in \mathbb{R} : -1 < x \le 1\}$ or $(-1,1]$

   c. $\bar{A}$; ***Answer:*** $\{x \in \mathbb{R} : (x < -2) \lor (x > 1)\}$

   d. $\bar{A} \cap \bar{B}$; ***Answer:*** $\{x \in \mathbb{R} : (x < -2) \lor (x \ge 3)\}$

   e. $A \setminus B$. ***Answer:*** $\{x \in \mathbb{R} : -2 \le x \le -1\}$ or $[-2, -1]$

3. (AY2022/23 Sem2 mid-term test.)
   For each of the following statements, prove whether it is true or false.

   a. There exist non-empty finite sets A and B such that $|A \cup B| = |A| + |B|$.

   b. There exist non-empty finite sets A and B such that $|A \cup B| \ne |A| + |B|$.

   c. There exist finite sets A and B such that $A \times \mathcal{P}(B) = \mathcal{P}(A \times B)$.

   Aiken thought that (c) is false and he provided a proof as follows:

   1. Let $|A| = n, |B| = k$.
   2. Then $|\mathcal{P}(B)| = 2^k$ (cardinality of power set).
   3. $|A \times B| = nk$ (cardinality of Cartesian product) and hence $|\mathcal{P}(A \times B)| = 2^{nk}$ (cardinality of power set).
   4. Then $|A \times \mathcal{P}(B)| = n2^k$ (cardinality of Cartesian product).
   5. Since $|A \times \mathcal{P}(B)| = n2^k \ne 2^{nk} = |\mathcal{P}(A \times B)|$, therefore the statement is false.

   Is Aiken's conclusion that the statement is false correct? Is his proof correct? If his proof is not correct, can you write a correct proof?

   Answers:

   a. True.
   One example: $A = \{a\}, B = \{b\}$. Then $|A \cup B| = |\{a, b\}| = 2 = 1 + 1 = |A| + |B|$.

   b. True.
   One example: $A = \{a, c\}, B = \{b, c\}$. Then $|A \cup B| = |\{a, b, c\}| = 3 \ne 4 = 2 + 2 = |A| + |B|$.

c. Aiken's conclusion is correct, but his proof is not.
If (i) $n = 1$, or (ii) $n = 2$ and $k = 1$, then $n2^k = 2^{nk}$, so line 5 is incorrect.

Proof:
1. Members of $A \times \mathcal{P}(B)$ are ordered pairs.
2. Members of $\mathcal{P}(A \times B)$ are sets. Note that $\mathcal{P}(A \times B)$ cannot be an empty set.
3. Ordered pairs and sets are different mathematical objects, and since $\mathcal{P}(A \times B)$ cannot be an empty set, hence $A \times \mathcal{P}(B) \neq \mathcal{P}(A \times B)$.

To think about:
- Why in line 2 we claim that $\mathcal{P}(A \times B)$ cannot be an empty set?
- Why would this proof require that $\mathcal{P}(A \times B)$ cannot be an empty set?

4. Let $A = \{2n + 1 : n \in \mathbb{Z}\}$ and $B = \{2n - 5 : n \in \mathbb{Z}\}$. Is $A = B$? Prove that your answer is correct. What does this tell us about how odd numbers may be defined?

Answer:
Yes, $A = B$. Proof as shown below. (Recall that: $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$.)

1. ($\subseteq$)
   1.1. Let $a \in A$.
   1.2. Use the definition of A to find an integer n such that $a = 2n + 1$.
   1.3. Then $a = 2n + 1 = 2(n + 3) - 5$.
   1.4. $n + 3 \in \mathbb{Z}$ (by closure of integers under $+$).
   1.5. Therefore, $a \in B$ (by the definition of B).
2. ($\supseteq$)
   2.1. Let $b \in B$.
   2.2. Use the definition of B to find an integer n such that $b = 2n - 5$.
   2.3. Then $b = 2n - 5 = 2(n - 3) + 1$.
   2.4. $n - 3 \in \mathbb{Z}$ (by closure of integers under $-$).
   2.5. Therefore, $b \in A$ (by the definition of A).
3. Therefore, $A = B$ (by the definition of set equality).

We use this definition of odd integers: An integer n is odd if and only if $n = 2k + 1$ for some integer k. The above tells us that we may define the set of odd numbers in many other ways.

Note: The above shows the general way of proving equality of 2 sets (i.e. prove that one is a subset of the other, and the other is a subset of the first). Sometimes, we may make use of the various definitions of set concepts, and set identities to prove set equality, examples are Q5 and Q6 below.

For this question, if one writes this proof:

1.	$A = \{2n + 1 : n \in \mathbb{Z}\}$	
2.	$= \{2(n + 3) - 5 : n \in \mathbb{Z}\}$	(by basic algebra)
3.	$= \{2k - 5 : k \in \mathbb{Z}\}$	(as $(n + 3) \in \mathbb{Z}$ by closure of integers under $+$)
4.	$= B$.	

The advantage of this proof seems to be its brevity. However, it is not entirely clear how elements in B can be expressed in the form of A. The issue lies in the change of variables from line 2 to line 3. So, for this question, it is better to use the general approach of proving that one is the subset of the other, and vice versa.

5. Using definitions of set operations (also called the **element method**), prove that for all sets A, B, C,

$$A \cap (B \setminus C) = (A \cap B) \setminus C.$$

Answer:

1. $A \cap (B \setminus C) = \{x : x \in A \land x \in (B \setminus C)\}$ — by the definition of $\cap$
2. $= \{x : x \in A \land (x \in B \land x \notin C)\}$ — by the definition of $\setminus$
3. $= \{x : (x \in A \land x \in B) \land x \notin C\}$ — by the associativity of $\land$
4. $= \{x : (x \in A \cap B) \land x \notin C\}$ — by the definition of $\cap$
5. $= (A \cap B) \setminus C$ — by the definition of $\setminus$

6. (Past year's midterm test question.)
Using **set identities** (Theorem 6.2.2), prove that for all sets A, B and C,

$$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C).$$

Answer:

$A \setminus (B \setminus C)$

1. $= A \setminus (B \cap \bar{C})$ — by the Set Difference Law
2. $= A \cap \overline{(B \cap \bar{C})}$ — by the Set Difference Law
3. $= A \cap (\bar{B} \cup \bar{\bar{C}})$ — by De Morgan's Law
4. $= A \cap (\bar{B} \cup C)$ — by the Double Complement Law
5. $= (A \cap \bar{B}) \cup (A \cap C)$ — by the Distributive Law
6. $= (A \setminus B) \cup (A \cap C)$ — by the Set Difference Law

7. For sets A and B, define $A \oplus B = (A \setminus B) \cup (B \setminus A)$.

   a. Let $A = \{1,4,9,16\}$ and $B = \{2,4,6,8,10,12,14,16\}$. Find $A \oplus B$.

   b. Using **set identities** (Theorem 6.2.2), prove that for all sets A and B,

$$A \oplus B = (A \cup B) \setminus (A \cap B).$$

Answers:

a. $A \setminus B = \{1,9\}$; $B \setminus A = \{2,6,8,10,12,14\}$; $A \oplus B = \{1,2,6,8,9,10,12,14\}$.

b.

1. $A \oplus B$
2. $= (A \setminus B) \cup (B \setminus A)$ — by the definition of $\oplus$
3. $= \big((A \cap \bar{B}) \cup (B \cap \bar{A})\big)$ — by the Set Difference Law
4. $= \big((A \cap \bar{B}) \cup B\big) \cap \big((A \cap \bar{B}) \cup \bar{A}\big)$ — by the Distributive Law
5. $= \big(B \cup (A \cap \bar{B})\big) \cap \big(\bar{A} \cup (A \cap \bar{B})\big)$ — by the Commutative Law
6. $= \big((B \cup A) \cap (B \cup \bar{B})\big) \cap \big((\bar{A} \cup A) \cap (\bar{A} \cup \bar{B})\big)$ — by the Distributive Law
7. $= \big((B \cup A) \cap (B \cup \bar{B})\big) \cap \big((A \cup \bar{A}) \cap (\bar{A} \cup \bar{B})\big)$ — by the Commutative Law
8. $= \big((B \cup A) \cap U\big) \cap \big(U \cap (\bar{A} \cup \bar{B})\big)$ — by the Complement Law
9. $= \big((A \cup B) \cap U\big) \cap \big((\bar{A} \cup \bar{B}) \cap U\big)$ — by the Commutative Law
10. $= (A \cup B) \cap (\bar{A} \cup \bar{B})$ — by the Identity Law
11. $= (A \cup B) \cap \overline{(A \cap B)}$ — by De Morgan's Law
12. $= (A \cup B) \setminus (A \cap B)$ — by the Set Difference Law

8. Let A and B be set. Show that $A \subseteq B$ if and only if $A \cup B = B$.

Answer:

1. (⇒)
   1.1. Suppose $A \subseteq B$. (Note: To show $A \cup B = B$, we need to show $A \cup B \subseteq B$ and $B \subseteq A \cup B$.)
   1.2. (To show $A \cup B \subseteq B$)
      1.2.1. Let $z \in A \cup B$.
      1.2.2. Then $z \in A$ or $z \in B$ (by the definition of ∪).
      1.2.3. Case 1: Suppose $z \in A$, then $z \in B$ as $A \subseteq B$ (from line 1.1).
      1.2.4. Case 2: Suppose $z \in B$, then $z \in B$.
      1.2.5. In either case, we have $z \in B$.
   1.3. (To show $A \cup B \supseteq B$)
      1.3.1. Let $z \in B$.
      1.3.2. Then $z \in A$ or $z \in B$ (by generalization).
      1.3.3. So $z \in A \cup B$ (by the definition of ∪).
   1.4. Therefore, $A \cup B = B$ (by the definition of set equality).

   Note: For 1.3, alternatively you may quote Theorem 6.2.1 Inclusion in Union (Lecture 5 slide 39).
2. (⇐)
   2.1. Suppose $A \cup B = B$.
   2.2. Let $z \in A$.
      2.2.1. Then $z \in A$ or $z \in B$ (by generalization).
      2.2.2. So $z \in A \cup B$ (by the definition of ∪).
      2.2.3. So $z \in B$ since $A \cup B = B$ (from line 2.1).
   2.3. Therefore, $A \subseteq B$.

   > To students:
   > Beware of skipping 2.2.1 and go directly from $z \in A$ to $z \in A \cup B$.

3. Therefore, $A \subseteq B$ if and only if $A \cup B = B$ (from 1 and 2).

9. Consider the claim:

   For all sets A, B, C, $\quad (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

   Aiken wrote the following proof:

   1. $x \in (A \setminus B) \cup (B \setminus A)$
   2. $\Rightarrow x \in (A \setminus B)$ or $x \in (B \setminus A)$
   3. $\Rightarrow x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$
   4. $\Rightarrow x \in A$ or $x \in B$ and $x \notin A$ and $x \notin B$
   5. $\Rightarrow x \in (A \cup B)$ and $x \in (\overline{A \cap B})$
   6. $\Rightarrow x \in (A \cup B) \cap (\overline{A \cap B})$
   7. $\Rightarrow x \in (A \cup B) \setminus (A \cap B)$.

   Therefore, $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

   (a) What is wrong with Aiken's proof?

   (b) Prove or disprove the claim.

Answers:

(a) Proof is only in one direction.

1. $x \in (A \setminus B) \cup (B \setminus A)$
2. $\Rightarrow x \in (A \setminus B)$ or $x \in (B \setminus A)$
3. $\Rightarrow x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$ — Ambiguous statement!
4. $\Rightarrow x \in A$ or $x \in B$ and $x \notin A$ and $x \notin B$ — Cannot commutate!
5. $\Rightarrow x \in (A \cup B)$ and $x \in (\overline{A \cap B})$ — $\bar{A} \cap \bar{B} \neq \overline{A \cap B}$!
6. $\Rightarrow x \in (A \cup B) \cap (\overline{A \cap B})$
7. $\Rightarrow x \in (A \cup B) \setminus (A \cap B)$.

(b)

1.	$(A \setminus B) \cup (B \setminus A)$	
2.	$= (A \cap \bar{B}) \cup (B \cap \bar{A})$	by the definition of \
3.	$= \big((A \cap \bar{B}) \cup B\big) \cap \big((A \cap \bar{B}) \cup \bar{A}\big)$	by the distributive law
4.	$= \big(B \cup (A \cap \bar{B})\big) \cap \big(\bar{A} \cup (A \cap \bar{B})\big)$	by the commutative law (x2)
5.	$= \big((B \cup A) \cap (B \cup \bar{B})\big) \cap \big((\bar{A} \cup A) \cap (\bar{A} \cup \bar{B})\big)$	by the distributive law (x2)
6.	$= \big((B \cup A) \cap U\big) \cap \big((\bar{A} \cup A) \cap (\bar{A} \cup \bar{B})\big)$	by the complement law
7.	$= (B \cup A) \cap \big((\bar{A} \cup A) \cap (\bar{A} \cup \bar{B})\big)$	by the identity law
8.	$= (A \cup B) \cap \big((\bar{A} \cup \bar{B}) \cap (A \cup \bar{A})\big)$	by the commutative law (x3)
9.	$= (A \cup B) \cap \big((\bar{A} \cup \bar{B}) \cap U\big)$	by the complement law
10.	$= (A \cup B) \cap (\bar{A} \cup \bar{B})$	by the identity law
11.	$= (A \cup B) \cap (\overline{A \cap B})$	by De Morgan's law
12.	$= (A \cup B) \setminus (A \cap B)$.	by the definition of \

(I didn't realise that Q9b is actually the same as Q7.)

10. Hogwarts School of Witchcraft and Wizardry where Harry Potter attended was divided into 4 houses: Gryffindor, Hufflepuff, Ravenclaw and Slytherin.

Let $HSWW$ be the set of students in the Hogwarts School of Witchcraft and Wizardry, and G, H, R and S be the sets of students in the 4 houses.

What are the necessary conditions for $\{G, H, R, S\}$ to be a partition of $HSWW$? Explain in English and the write logical statements.

Answers:

The necessary conditions are every student is in exactly one of the four houses, and every house has at least one student.

$G \cap H = G \cap R = G \cap S = H \cap R = H \cap S = R \cap S = \emptyset$.
(That is, the houses are mutually disjoint sets.)

$G \cup H \cup R \cup S = HSWW$. (That is, every Hogwarts student is in one of the houses.)

$G \neq \emptyset \wedge H \neq \emptyset \wedge R \neq \emptyset \wedge S \neq \emptyset$. (That is, every house has at least one student.)

For questions 11 and 12, for sets $A_m, A_{m+1}, \dots, A_n$, we define the following:

$$\bigcup_{i=m}^{n} A_i = A_m \cup A_{m+1} \cup \dots \cup A_n$$

and

$$\bigcap_{i=m}^{n} A_i = A_m \cap A_{m+1} \cap \dots \cap A_n$$

11. Let $A_i = \{x \in \mathbb{Z} : i \leq x \leq 2i\}$ for all integers i. Write A_{-2}, $\bigcup_{i=3}^{5} A_i$ and $\bigcap_{i=3}^{5} A_i$ in set-roster notation, set-builder notation, and interval notation.

    Answers:

    $$A_{-2} = \emptyset$$

    $$\bigcup_{i=3}^{5} A_i = \{3,4,5,6,7,8,9,10\} = \{x \in \mathbb{Z} : 3 \leq x \leq 10\} = [3,10].$$

    $$\bigcap_{i=3}^{5} A_i = \{5,6\} = \{x \in \mathbb{Z} : 5 \leq x \leq 6\} = [5,6].$$

12. Let $V_i = \left\{x \in \mathbb{R} : -\frac{1}{i} \leq x \leq \frac{1}{i}\right\} = \left[-\frac{1}{i}, \frac{1}{i}\right]$ for all positive integers i.

    a. What is $\bigcup_{i=1}^{4} V_i$? ***Answer:*** $\bigcup_{i=1}^{4} V_i = [-1,1]$.

    b. What is $\bigcap_{i=1}^{4} V_i$? ***Answer:*** $\bigcap_{i=1}^{4} V_i = \left[-\frac{1}{4}, \frac{1}{4}\right]$.

    c. What is $\bigcup_{i=1}^{n} V_i$, where n is a positive integer? ***Answer:*** $\bigcup_{i=1}^{n} V_i = [-1,1]$.

    d. What is $\bigcap_{i=1}^{n} V_i$, where n is a positive integer? ***Answer:*** $\bigcap_{i=1}^{n} V_i = \left[-\frac{1}{n}, \frac{1}{n}\right]$.

    e. Are $V_1, V_2, V_3, \dots$ mutually disjoint?

       Answer: $V_1, V_2, V_3, \dots$ are not mutually disjoint. They have the common element 0.

       $$V_1 = [-1,1];\ V_2 = \left[-\frac{1}{2}, \frac{1}{2}\right];\ V_3 = \left[-\frac{1}{3}, \frac{1}{3}\right];\ V_4 = \left[-\frac{1}{4}, \frac{1}{4}\right]; \cdots; V_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$$