CS1231S: Discrete Structures

Tutorial #3: Sets

(Week 5: 9 – 13 September 2024)

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1. Discussion Questions

These are meant for your own discussion. No answers will be provided.

D1 Which of the following are true? Which are false?

(a) $\emptyset \in \emptyset$.

(b) $\emptyset \subseteq \emptyset$.

(c) $\emptyset \in \{\emptyset\}$.

(d) $\emptyset \subseteq \{\emptyset\}$.

(e) $\{\emptyset, 1\} = \{1\}$.

(f) $1 \in \{\{1,2\},\{2,3\},4\}$.

(g) $\{1,2\} \subseteq \{3,2,1\}$.

(h) $\{3,3,2\} \subsetneq \{3,2,1\}$.

(i) $\emptyset \in \{1,2\}$.

(j) $\{1,2,3\} = \{3,1,1,2\}$.

(k) $\{5\} \in \{2,5\}$.

(l) $\{1,2\} \in \{1,\{2,1\}\}$.

D2. Let $A = \{0,1,4,5,6,9\}$ and $B = \{0,2,4,6,8\}$. Find $|A \cap B|$ and $|A \cup B|$.

D3. Let $A = \{1,\{1,2\},2,\{2,1,1\}\}$. Find $|A|$ and write out $A \times A$ in its simplest form (that is, with no duplicate elements).

2. Tutorial Questions

Note that the sets here are finite sets, unless otherwise stated.

1. Let $\mathcal{P}(A)$ denotes the power set of A. Find the following:

   a. $\mathcal{P}(\{a,b,c\})$;

   b. $\mathcal{P}\left(\mathcal{P}(\mathcal{P}(\emptyset))\right)$.

2. Let the universal set be $\mathbb{R}$, $A = \{x \in \mathbb{R} : -2 \le x \le 1\}$ and $B = \{x \in \mathbb{R} : -1 < x < 3\}$.

   Note that *A* and *B* may be written as [-2,1] and (-1,3) respectively using real numbers interval notation (Lecture #5 slide 27). Find the following and write your answers in set-builder notation, and if appropriate, also in interval notation.

   a. $A \cup B$;

   b. $A \cap B$;

   c. $\bar{A}$;

   d. $\bar{A} \cap \bar{B}$;

   e. $A \setminus B$.

3. (AY2022/23 Sem2 mid-term test.)
   For each of the following statements, prove whether it is true or false.

   a. There exist non-empty finite sets A and B such that $|A \cup B| = |A| + |B|$.

   b. There exist non-empty finite sets A and B such that $|A \cup B| \ne |A| + |B|$.

   c. There exist finite sets A and B such that $A \times \mathcal{P}(B) = \mathcal{P}(A \times B)$.

Aiken thought that (c) is false and he provided a proof as follows:

1. Let $|A| = n, |B| = k$.
2. Then $|\mathcal{P}(B)| = 2^k$ (cardinality of power set).
3. $|A \times B| = nk$ (cardinality of Cartesian product) and hence $|\mathcal{P}(A \times B)| = 2^{nk}$ (cardinality of power set).
4. Then $|A \times \mathcal{P}(B)| = n2^k$ (cardinality of Cartesian product).
5. Since $|A \times \mathcal{P}(B)| = n2^k \neq 2^{nk} = |\mathcal{P}(A \times B)|$, therefore the statement is false.

Is Aiken's conclusion that the statement is false correct? Is his proof correct? If his proof is not correct, can you write a correct proof?

4. Let $A = \{2n + 1 : n \in \mathbb{Z}\}$ and $B = \{2n - 5 : n \in \mathbb{Z}\}$. Is $A = B$? Prove that your answer is correct. What does this tell us about how odd numbers may be defined?

5. Using definitions of set operations (also called the **element method**), prove that for all sets A, B, C,

$$A \cap (B \setminus C) = (A \cap B) \setminus C.$$

6. (Past year's midterm test question.)
Using **set identities** (Theorem 6.2.2), prove that for all sets A, B and C,

$$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C).$$

7. For sets A and B, define $A \oplus B = (A \setminus B) \cup (B \setminus A)$.

   a. Let $A = \{1,4,9,16\}$ and $B = \{2,4,6,8,10,12,14,16\}$. Find $A \oplus B$.

   b. Using **set identities** (Theorem 6.2.2), prove that for all sets A and B,

$$A \oplus B = (A \cup B) \setminus (A \cap B).$$

8. Let A and B be set. Show that $A \subseteq B$ if and only if $A \cup B = B$.

9. Consider the claim:

For all sets A, B, C, $\quad (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Aiken wrote the following proof:

1. $x \in (A \setminus B) \cup (B \setminus A)$
2. $\Rightarrow x \in (A \setminus B)$ or $x \in (B \setminus A)$
3. $\Rightarrow x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$
4. $\Rightarrow x \in A$ or $x \in B$ and $x \notin A$ and $x \notin B$
5. $\Rightarrow x \in (A \cup B)$ and $x \in (\overline{A \cap B})$
6. $\Rightarrow x \in (A \cup B) \cap (\overline{A \cap B})$
7. $\Rightarrow x \in (A \cup B) \setminus (A \cap B)$.

Therefore, $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

(a) What is wrong with Aiken's proof?

(b) Prove or disprove the claim.

10. Hogwarts School of Witchcraft and Wizardry where Harry Potter attended was divided into 4 houses: Gryffindor, Hufflepuff, Ravenclaw and Slytherin.

    Let $HSWW$ be the set of students in the Hogwarts School of Witchcraft and Wizardry, and G, H, R and S be the sets of students in the 4 houses.

    What are the necessary conditions for $\{G, H, R, S\}$ to be a partition of $HSWW$? Explain in English and the write logical statements.

For questions 11 and 12, for sets $A_m, A_{m+1}, \ldots, A_n$, we define the following:

$$\bigcup_{i=m}^{n} A_i = A_m \cup A_{m+1} \cup \ldots \cup A_n$$

and

$$\bigcap_{i=m}^{n} A_i = A_m \cap A_{m+1} \cap \ldots \cap A_n$$

11. Let $A_i = \{x \in \mathbb{Z} : i \leq x \leq 2i\}$ for all integers i. Write A_{-2}, $\bigcup_{i=3}^{5} A_i$ and $\bigcap_{i=3}^{5} A_i$ in set-roster notation, set-builder notation, and interval notation.

12. Let $V_i = \left\{x \in \mathbb{R} : -\frac{1}{i} \leq x \leq \frac{1}{i}\right\} = \left[-\frac{1}{i}, \frac{1}{i}\right]$ for all positive integers i.

    a. What is $\bigcup_{i=1}^{4} V_i$?

    b. What is $\bigcap_{i=1}^{4} V_i$?

    c. What is $\bigcup_{i=1}^{n} V_i$, where n is a positive integer?

    d. What is $\bigcap_{i=1}^{n} V_i$, where n is a positive integer?

    e. Are $V_1, V_2, V_3, \ldots$ mutually disjoint?