CS1231S: Discrete Structures

Tutorial #4: Relations & Equivalence Relations

(Week 6: 16 – 20 September 2024)

Answers

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1. Let $A = \{1,2,\ldots,10\}$ and $B = \{2,4,6,8,10,12,14\}$. Define a relation R from A to B by setting

$$x\ R\ y \ \Leftrightarrow x \text{ is prime and } x \mid y$$

for each $x \in A$ and each $y \in B$. Write down the sets R and R^{-1} in **roster notation**. Do not use ellipses (…) in your answers.

Answers:

$R \quad = \{(2,2),(2,4),(2,6),(2,8),(2,10),(2,12),(2,14),(3,6),(3,12),(5,10),(7,14)\}.$

$R^{-1} = \{(2,2),(4,2),(6,2),(8,2),(10,2),(12,2),(14,2),(6,3),(12,3),(10,5),(14,7)\}.$

2. Let R be a relation on a set A. Show that the following are logically equivalent by using this strategy: (i) implies (ii), (ii) implies (iii), and (iii) implies (i).

(i) R is symmetric, i.e. $\forall x, y \in A\ (x\ R\ y\ \Rightarrow y\ R\ x)$.

(ii) $\forall x, y \in A\ (x\ R\ y\ \Leftrightarrow y\ R\ x)$.

(iii) $R = R^{-1}$.

Answer:

1. ((i) ⇒ (ii))
   1.1. Suppose R is symmetric.
   1.2. Let $x, y \in A$.
   1.3. (⇒) If $x\ R\ y$, then $y\ R\ x$ by the symmetry of R.
   1.4. (⇐) If $y\ R\ x$, then $x\ R\ y$ by the symmetry of R.
   1.5. From 1.3 and 1.4, we have $x\ R\ y\ \Leftrightarrow y\ R\ x$.

2. ((ii) ⇒ (iii))
   2.1. Suppose $\forall x, y \in A\ (x\ R\ y\ \Leftrightarrow y\ R\ x)$.
   2.2. Then for all $x, y \in A$,

2.2.1.	$(x, y) \in R$	$\Leftrightarrow$	$x\ R\ y$	by the definition of $x\ R\ y$
2.2.2.		$\Leftrightarrow$	$y\ R\ x$	by 2.1
2.2.3.		$\Leftrightarrow$	$x\ R^{-1}\ y$	by the definition of R^{-1}
2.2.4.		$\Leftrightarrow$	$(x, y) \in R^{-1}$	by the definition of $x\ R^{-1}\ y$.

   2.3. Hence $R = R^{-1}$.

3. ((iii) ⇒ (i))
   3.1. Suppose $R = R^{-1}$.
      3.1.1. Let $x, y \in A$ such that $x\ R\ y$.
      3.1.2. Then $x\ R^{-1}\ y$ as $R = R^{-1}$.
      3.1.3. $\therefore y\ R\ x$ by the definition of R^{-1}
   3.2. Hence R is symmetric.

4. Therefore (i), (ii) and (iii) are logically equivalent.

3. For each of the relations defined below, determine whether it is (i) reflexive, (ii) symmetric, (iii) transitive, and (iv) an equivalence relation. If a property is false for the relation, give a counter-example.

   (a) Let $A = \{1,2,3\}$, $Q = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$, where Q is a relation on A.

   (b) Define the relation E on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}, x\ E\ y \Leftrightarrow x = y$.

   (c) Define the relation R on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}, x\ R\ y \Leftrightarrow xy \geq 0$.

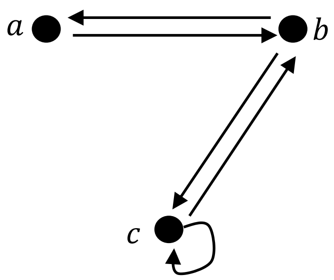
   (d) Define the relation S on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}, x\ S\ y \Leftrightarrow xy > 0$.

   (e) Define the relation T on $\mathbb{Z}$ by setting, for all $x, y \in \mathbb{Z}, x\ T\ y \Leftrightarrow -2 \leq x - y \leq 2$.

Answers:

	Reflexive?	Symmetric?	Transitive?	Equivalence relation?
Q	Yes	No $1\ Q\ 2$ but $2 \not Q\ 1$	Yes	No
E	Yes	Yes	Yes	Yes
R	Yes	Yes	No $1\ R\ 0$ and $0\ R - 1$ but $1 \not R - 1$	No
S	No $0 \not S\ 0$	Yes	Yes	No
T	Yes	Yes	No $-2\ T\ 0$ and $0\ T 2$ but $-2 \not T\ 2$	No

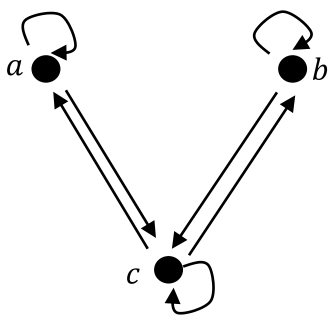
4. The directed graph of a binary relation R on a set $A = \{a, b, c\}$ is shown below.



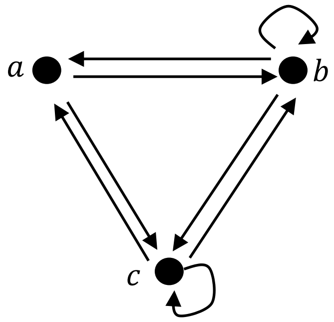
Draw the directed graph for each of the following and determine if it is transitive or not. If it is not transitive, explain.

(a) $R \circ R$ (b) $R \circ R \circ R$ (c) $(R \circ R) \cup R$

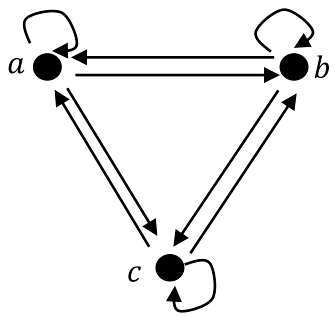
Answers:



(a) $R \circ R$



$(b)\ R \circ R \circ R$



(c) $(R \circ R) \cup R$

An easy way to compute $R \circ R$ is as follows: (i) Start with the first element a and trace all possible destinations after taking exactly two arrows (the same arrow may be taken twice). Then in the resulting graph, draw an arrow from a to all such destinations; (ii) Repeat for elements b and c.

To compute $R \circ R \circ R$, use the same method as above, but take exactly three arrows.

(a) $R \circ R$: Not transitive. Reason: $a(R \circ R)c \wedge c(R \circ R)b$ but $a(\cancel{R \circ R})b.$

(b) $R \circ R \circ R$: Not transitive. Reason: $a(R \circ R \circ R)c \wedge c(R \circ R \circ R)a$ but $a(\cancel{R \circ R \circ R})a.$

(c) $(R \circ R) \cup R$: Transitive.

5. (AY2023/24 semester 1 midterm test).
Which of the following are true for all equivalence relations R?

(a) $R^{-1} \circ R = R \circ R^{-1}$

(b) $R \subseteq R \circ R$

(c) $R \circ R \subseteq R$

(d) $R \circ R^{-1} = R$

Answers:

(a) True, because equivalence relations are symmetric. R is symmetric if and only if $R = R^{-1}$ (by Q2). Proof: $R^{-1} \circ R = R \circ R = R \circ R^{-1}$.

(b) True, because equivalence relations are reflexive. Let R be a relation on the set A. Proof:
   1. Let $(x, y) \in R$, where $x, y \in A$.
   2. Since $(x, x), (y, y) \in R$ (by reflexivity of R), composing (x, x) with (x, y), or (x, y) with (y, y), we have $(x, y) \in R \circ R$ (by definition of composition of relations).

(c) True, because equivalence relations are transitive. Let R be a relation on the set A. Proof:
   1. Let $(x, z) \in R \circ R$, where $x, z \in A$.
   2. There exists some $y \in A$ such that $(x, y) \in R$ and $(y, z) \in R$ (by definition of composition).
   3. Hence by transitivity of R, $(x, z) \in R$.

(d) True. Proof:
   1. As R is symmetric, $R^{-1} = R$ by Q2.
   2. By (b) and (c), $R \circ R = R$.
   3. Therefore, $R \circ R^{-1} = R \circ R = R$ by lines 1 and 2.

6. (AY2023/24 semester 1 exam).
Define the following relation on $A = \{1,2,3\}$:

$$R = \{ (1,1), (1,2), (2,1), (2,2), (3,3) \}.$$

Find $R \circ R \circ R \circ R \circ R \circ R \circ R$.

(How do you make use of some question above to get the answer quickly?)

Answer:

$\{ (1,1), (1,2), (2,1), (2,2), (3,3) \}$.

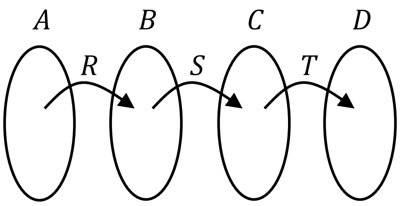
Note that R is an equivalence relation. From Q5(d), we have $R \circ R = R$.

Hence, by associativity of composition of relations, we have

$$R \circ R \circ R \circ R \circ R \circ R \circ R = \big((R \circ R) \circ (R \circ R)\big) \circ \big((R \circ R) \circ R\big) = (R \circ R) \circ (R \circ R) = R \circ R = R.$$

7. Let A, B, C, D be sets and $R \subseteq A \times B, S \subseteq B \times C$, and $T \subseteq C \times D$. Prove that

$$T \circ (S \circ R) = (T \circ S) \circ R.$$



That is, composition of relations is associative.

Answer:

1. Note that $S \circ R \subseteq A \times C$ and $T \circ S \subseteq B \times D$.
2. ($\subseteq$) Suppose $(a, d) \in T \circ (S \circ R)$
   2.1. Then there is a $c \in C$ such that $(a, c) \in S \circ R$ and $(c, d) \in T$.
   (by the definition of composition of relations)
   2.2. Moreover, from $(a, c) \in S \circ R$ there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.
   2.3. From $(b, c) \in S$ in 2.2 and $(c, d) \in T$ in 2.1, we have $(b, d) \in T \circ S$.
   2.4. From $(a, b) \in R$ in 2.2 and $(b, d) \in T \circ S$ in 2.3, we have $(a, d) \in (T \circ S) \circ R$.
   2.5. Therefore, $T \circ (S \circ R) \subseteq (T \circ S) \circ R$.
3. ($\supseteq$) Suppose $(a, d) \in (T \circ S) \circ R$
   3.1. Then there is a $b \in B$ such that $(a, b) \in R$ and $(b, d) \in T \circ S$.
   (by the definition of composition of relations)
   3.2. Moreover, from $(b, d) \in T \circ S$ there is a $c \in C$ such that $(b, c) \in S$ and $(c, d) \in T$.
   3.3. From $(a, b) \in R$ in 3.1 and $(b, c) \in S$ in 3.2, we have $(a, c) \in S \circ R$.
   3.4. From $(a, c) \in S \circ R$ in 3.3 and $(c, d) \in T$ in 3.2, we have $(a, d) \in T \circ (S \circ R)$.
   3.5. Therefore,$(T \circ S) \circ R \subseteq T \circ (S \circ R)$.
4. Therefore, $T \circ (S \circ R) = (T \circ S) \circ R$.

8. (AY2020/21 Semester 1 exam question)
Define an equivalence relation $\sim$ on $\mathbb{Z}^+ \times \mathbb{Z}^+$ by setting, for all $a, b, c, d \in \mathbb{Z}^+$,

$$(a, b) \sim (c, d) \Leftrightarrow ab = cd.$$

Write down the equivalence classes $[(1,1)]$ and $[(4,3)]$ in **roster notation**.

Answers:

$[(1,1)] = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : (1,1) \sim (x, y)\}$ by the definition of equivalence class
$\quad = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : 1 \times 1\ (= 1) = ab\}$ by the definition of $\sim$
$\quad = \{(1,1)\}$.

$[(4,3)] = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : (4,3) \sim (x, y)\}$ by the definition of equivalence class
$\quad = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : 4 \times 3\ (= 12) = ab\}$ by the definition of $\sim$
$\quad = \{(1,12), (2,6), (3,4), (4,3), (6,2), (12,1)\}$.

9. Consider the relation $S = \{(m, n) \in \mathbb{Z}^2 : m^3 + n^3 \text{ is even}\}$. (Recall that $\mathbb{Z}^2$ means $\mathbb{Z} \times \mathbb{Z}$.) Determine (a) S^{-1}, (b) $S \circ S$ and (c) $S \circ S^{-1}$.

You may use theorems involving the sum of even and odd integers without quoting them (eg: the sum of two even integers is even; the sum of an even integer and odd integer is odd; etc.).

Answers:

(a) $S^{-1} = \{(x, y) \in \mathbb{Z}^2 : (y, x) \in S\}$ by the definition of inverse relation
$\quad = \{(x, y) \in \mathbb{Z}^2 : y^3 + x^3 \text{ is even}\}$ by the definition of S
$\quad = \{(x, y) \in \mathbb{Z}^2 : x^3 + y^3 \text{ is even}\}$ by the commutative law of addition
$\quad = S$ by the definition of S

(b) $S \circ S = S$
Proof:
1. ($\subseteq$) Suppose $(x, z) \in S \circ S$
   1.1. Then $(x, y) \in S$ and $(y, z) \in S$ for some $y \in \mathbb{Z}$.
   (by the definition of composition of relations)
   1.2. So $x^3 + y^3$ is even and $y^3 + z^3$ is even.
   1.3. This implies that $x^3 + 2y^3 + z^3$ is even.
   1.4. This implies that $x^3 + z^3$ is even as $2y^3$ is even.
   1.5. Therefore, $(x, z) \in S$ by the definition of S.
2. ($\supseteq$) Suppose $(x, z) \in S$
   2.1. Then $x^3 + z^3$ is even by the definition of S.
   2.2. Case 1: x^3 is odd.
      2.2.1. Then z^3 is also odd.
      2.2.2. This implies $x^3 + 1^3$ is even and $1^3 + z^3$ is even.
      2.2.3. Thus $(x, 1) \in S$ and $(1, z) \in S$ by the definition of S.
      2.2.4. So $(x, z) \in S \circ S$ by the definition of composition of relations.
   2.3. Case 2: x^3 is even.
      2.3.1. Then z^3 is also even.
      2.3.2. This implies $x^3 + 0^3$ is even and $0^3 + z^3$ is even.
      2.3.3. Thus $(x, 0) \in S$ and $(0, z) \in S$ by the definition of S.
      2.3.4. So $(x, z) \in S \circ S$ by the definition of composition of relations.
   2.4. In all cases, $(x, z) \in S \circ S$.
3. $\therefore S \circ S = S$.

Alternatively, for 2:

2. ($\supseteq$) Suppose $(x, z) \in S$
   2.1. Note that $(x, x) \in S$ as $x^3 + x^3$ is even.
   2.2. Since $(x, x) \in S$ and $(x, z) \in S$, we have $(x, z) \in S \circ S$ by the definition of composition of relations.
   2.3. Hence, $S \subseteq S \circ S$.

(c) It follows from (a) and (b) that $S \circ S^{-1} = S \circ S = S$.

10. Define a relation $\sim$ on $\mathbb{Z} \setminus \{0\}$ as follows: $\forall a, b \in \mathbb{Z} \setminus \{0\}\ (a{\sim}b \Leftrightarrow ab > 0)$.

    (a) Prove that $\sim$ is an equivalence relation. You may adopt the appropriate **order axioms** and **theorems** in *Appendix A: Properties of the Real Numbers* for the integers. (Appendix A is available on Canvas > Files as well as the CS1231S webpage at https://www.comp.nus.edu.sg/~cs1231s/2_resources/lectures.html.)

    (b) Determine all the distinct equivalence classes formed by this relation $\sim$.

Answers:

(a) Proof:

1. ("Reflexivity")
   1.1. Let $a \in \mathbb{Z} \setminus \{0\}$, since $a \neq 0$, we have $a^2 > 0$ by T21.
   1.2. Thus, $a{\sim}a$ by the definition of $\sim$.
   1.3. Hence $\sim$ is reflexive.

T21. If $a \neq 0$, then $a^2 > 0$.

2. ("Symmetry")
   2.1. For any $a, b \in \mathbb{Z} \setminus \{0\}$, if $a{\sim}b$, then $ab > 0$ by the definition of $\sim$.
   2.2. Then $ba > 0$ by the commutative law of multiplication.
   2.3. So $b{\sim}a$ by the definition of $\sim$.
   2.4. Hence $\sim$ is symmetric.
3. ("Transitivity")
   3.1. For any $a, b, c \in \mathbb{Z} \setminus \{0\}$, suppose $a{\sim}b$ and $b{\sim}c$.
   3.2. Then $ab > 0$ and $bc > 0$ by the definition of $\sim$.
   3.3. Multiplying ab with bc (both positive) gives $ab^2c > 0$ by Ord1.
   3.4. Then $(ac)b^2 > 0$ by the associative and commutative laws of multiplication.
   3.5. Then both (ac) and b^2 are positive, or both are negative, by T25.
   3.6. Since $b^2 > 0$ (by T21, as $b \neq 0$), (ac) must also be positive.
   3.7. Thus $a{\sim}c$ by the definition of $\sim$.
   3.8. Hence $\sim$ is transitive.

Ord1. If a and b are positive, so are $a + b$ and ab.

T25. If $ab > 0$, then both a and b are positive, or both are negative.

4. Therefore, $\sim$ is an equivalence relation.

(b) T25 states that if $ab > 0$, then both a and b are positive, or both are negative.

Thus, all positive integers are $\sim$-related to one another, and likewise, all negative integers are $\sim$-related to one another.

Therefore, the two distinct equivalence classes are: $\{a \in \mathbb{Z} - \{0\} : a > 0\}$ and $\{a \in \mathbb{Z} - \{0\} : a < 0\}$. Or, choosing 1 and -1 as representatives, the two equivalence classes are [1] and [-1].