CS1231S: Discrete Structures

Tutorial #5: Relations & Partial Orders

(Week 7: 30 September – 4 October 2024)

Answers

II. Tutorial Questions

1. Let S be the set of all strings over the alphabet $\mathcal{A} = \{s, u\}$, i.e. an element of S is a sequence of characters, each of which is either s or u. Examples of elements of S are: ε (the empty string), $s, u, sus, usssuu$, and $susussssu$.

   Define a relation R on S by the following: $\forall a, b \in S \ (aRb \ \Leftrightarrow len(a) \leq len(b))$ where $len(x)$ denotes the length of x, i.e. the number of characters in x.

   Is R a partial order? Prove or disprove it.

   > Let R be a relation on a set A.
   > R is **antisymmetric** iff $\forall x, y \in A \ (xRy \wedge yRx \Rightarrow x = y)$.

   Answer:
   R is not antisymmetric, hence it is not a partial order. Counterexample: $s \ R \ u$ and $u \ R \ s$ but $s \neq u$.

2. [AY2022/23 Semester 1 Midterm Test]
   Let $A = \{2,3,5,7,21,30,84,99\}$ and let $\preccurlyeq$ be a partial order on the set A defined by the "divides" relation, that is, $x \preccurlyeq y \ \Leftrightarrow x|y$. Which of the following statements are true?

   (i) The partial order has a linearization $\preccurlyeq^*$ such that $21 \preccurlyeq^* 7$.
   (ii) The partial order has a linearization $\preccurlyeq^*$ such that $3 \preccurlyeq^* 2$.
   (iii) The partial order has a linearization $\preccurlyeq^*$ such that $21 \preccurlyeq^* 5 \preccurlyeq^* 84$.
   (iv) The partial order has a linearization $\preccurlyeq^*$ such that $99 \preccurlyeq^* 84 \preccurlyeq^* 30$.

   Answer: Only (ii), (iii) and (iv) are true.

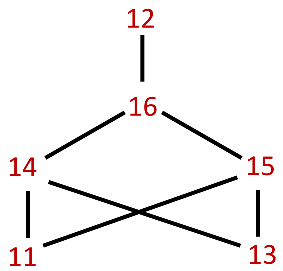
   2 and 3 are noncomparable, 5 and 21 are noncomparable, 30, 84 and 99 are noncomparable, so they can appear in any relative order in the linearization.

3. [AY2023/24 semester 1 Midterm Test]
   Let $A = \{11,12,13,14,15,16\}$. For each $x \in A$, define $F_x = \{k \in \mathbb{Z}^+ : k|x\}$, where | is the "divides" relation. Define also a partial order $\preccurlyeq$ on A by setting for all $x, z \in A$:

   $$x \preccurlyeq z \ \Leftrightarrow (F_x = F_z) \vee (|F_x| < |F_z|).$$

   What are the minimal, smallest, maximal, and largest elements of A with respect to $\preccurlyeq$?

   Answers:
   Minimal: 11, 13.
   Smallest: None.
   Maximal: 12.
   Largest: 12.



> Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and let $c \in A$.
> - c is a **maximal element** of A iff $\forall x \in A \ (c \preccurlyeq x \Rightarrow c = x)$.
> - c is a **minimal element** of A iff $\forall x \in A \ (x \preccurlyeq c \Rightarrow c = x)$.
> - c is the **largest element** of A iff $\forall x \in A \ (x \preccurlyeq c)$.
> - c is the **smallest element** of A iff $\forall x \in A \ (c \preccurlyeq x)$.

4. Given the partial order $\preccurlyeq$ on A in question 3 above, one of the linearizations $\preccurlyeq^*$ of $\preccurlyeq$ is shown below:

$$11 \preccurlyeq^* 13 \preccurlyeq^* 14 \preccurlyeq^* 15 \preccurlyeq^* 16 \preccurlyeq^* 12$$

Write out all the other possible linearizations $\preccurlyeq^*$.

Answers: There are 4 possible linearizations; the other 3 are:

$$11 \preccurlyeq^* 13 \preccurlyeq^* 15 \preccurlyeq^* 14 \preccurlyeq^* 16 \preccurlyeq^* 12$$
$$13 \preccurlyeq^* 11 \preccurlyeq^* 14 \preccurlyeq^* 15 \preccurlyeq^* 16 \preccurlyeq^* 12$$
$$13 \preccurlyeq^* 11 \preccurlyeq^* 15 \preccurlyeq^* 14 \preccurlyeq^* 16 \preccurlyeq^* 12$$

5. Let $\mathcal{P}(A)$ denote the power set of set A. Prove that the binary relation $\subseteq$ on $\mathcal{P}(A)$ is a partial order.

Answer:

Proof:

1. (Reflexivity) Take any $S \in \mathcal{P}(A)$,
   1.1. $S \subseteq S$ by the definition of subset.
   1.2. Hence $\subseteq$ is reflexive.
2. (Antisymmetry) Take any $S, T \in \mathcal{P}(A)$,
   2.1. Suppose $S \subseteq T$ and $T \subseteq S$.
   2.2. Then $S = T$ by the definition of set equality.
   2.3. Hence $\subseteq$ is antisymmetric.
3. (Transitivity)
   3.1. $\subseteq$ is transitive by Theorem 6.2.1.
4. Therefore $\subseteq$ on $\mathcal{P}(A)$ is a partial order.

Theorem 6.2.1
For all sets A, B, C,
$A \subseteq B \land B \subseteq C \Rightarrow A \subseteq C$.

6. Let $B = \{0,1\}$ and define the binary relation R on $B \times B$ as follows:

$$\forall (a,b),(c,d) \in B \times B \ \big((a,b)\ R\ (c,d) \Leftrightarrow (a \le c) \land (b \le d)\big).$$

(a) Prove that R is a partial order.
(b) Draw the Hasse diagram for R.
(c) Find the maximal, largest, minimal and smallest elements.
(d) Is $(B \times B, R)$ well-ordered?

Let R be a relation on a set A.
- R is **reflexive** iff $\forall x \in A\ (xRx)$.
- R is **antisymmetric** iff $\forall x, y \in A\ (xRy \land yRx \Rightarrow x = y)$.
- R is **transitive** iff $\forall x, y, z \in A\ (xRy \land yRz \Rightarrow xRz)$.

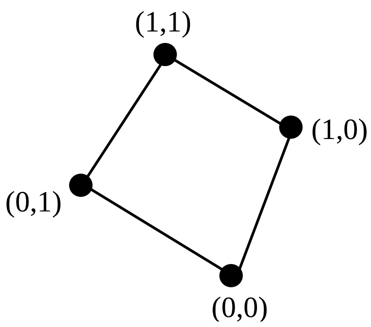
Answers:

(a) Proof:

1. (Reflexivity) Take any $(a,b) \in B \times B$,
   1.1. $a \le a$ and $b \le b$.
   1.2. So $(a,b)\ R\ (a,b)$ by the definition of R.
   1.3. Hence R is reflexive.
2. (Antisymmetry) Take any $(a,b),(c,d) \in B \times B$,
   2.1. Suppose $(a,b)\ R\ (c,d)$ and $(c,d)\ R\ (a,b)$.
   2.2. Then $a \le c, b \le d, c \le a$ and $d \le b$ by the definition of R.
   2.3. Then $a = c$ and $b = d$ by the antisymmetry of $\le$.
   2.4. So $(a,b) = (c,d)$ by equality of ordered pairs.
   2.5. Hence R is antisymmetric.

3. (Transitivity) Take any $(a, b), (c, d), (e, f) \in B \times B$,
   3.1. Suppose $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$.
   3.2. Then $a \leq c, b \leq d, c \leq e$ and $d \leq f$ by the definition of R.
   3.3. Then $a \leq e$ and $b \leq f$ by the transitivity of $\leq$.
   3.4. So $(a, b)\ R\ (e, f)$ by the definition of R.
   3.5. Hence R is transitive.
4. Therefore R on $B \times B$ is a partial order.

(b)



(c) Maximal and largest: (1,1);
Minimal and smallest: (0,0).

(d) No. It is not even a total order, because (0,1) and (1,0) are not comparable.

> R is a total order on a set A iff
> R is a partial order on A and $\forall x, y \in A\ (xRy \lor yRx)$.

7. Let R be a binary relation on a non-empty set A. Let $x, y \in A$. Define a relation S on A by

$$x\ S\ y \Leftrightarrow (x = y) \lor (x\ R\ y) \text{ for all } x, y \in A.$$

Show that:

(a) S is reflexive;

(b) $R \subseteq S$; and

(c) if S' is another reflexive relation on A and $R \subseteq S'$, then $S \subseteq S'$.

What is this relation S called? (Hint: Refer to Transitive Closure in Lecture 6).

Answers:

> Let R be a relation on a set A.
> R is **reflexive** iff $\forall x \in A\ (xRx)$.

(a) 1. Let $x \in A$.
2. $x = x$, so $x\ S\ x$ by the definition of S.
3. Therefore, S is reflexive.

(b) 1. Suppose $(x, y) \in R$, that is, $x\ R\ y$.
2. So $x\ S\ y$ by the definition of S.
3. So $(x, y) \in S$.
4. Therefore, $R \subseteq S$ by the definition of $\subseteq$.

(c) 1. Suppose $(x, y) \in S$.
2. Then $x\ S\ y$, which means $x = y \lor x\ R\ y$ by the definition of S.
3. Case 1: $x = y$
   3.1. Then $x\ S'\ y$ since S' is reflexive.
   3.2. So $(x, y) \in S'$.
4. Case 2: $x\ R\ y$
   4.1. Then $(x, y) \in R \subseteq S'$.
   4.2. Then $(x, y) \in S'$.
5. In all cases, $(x, y) \in S'$.
6. Therefore, $S \subseteq S'$.

The relation S is called the reflexive closure of R. It is the smallest relation on A that is reflexive and contains R as a subset.

8. Let R be a binary relation on a set A.
We have defined antisymmetry in class: R is **antisymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y).$$

We define asymmetry here. R is **asymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x).$$

(a) Find a binary relation on A that is both asymmetric and antisymmetric.

(b) Find a binary relation on A that is not asymmetric but antisymmetric.

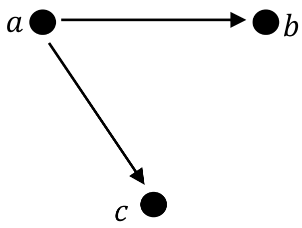
(c) Find a binary relation on A that is asymmetric but not antisymmetric.

(d) Find a binary relation on A that is neither asymmetric nor antisymmetric.

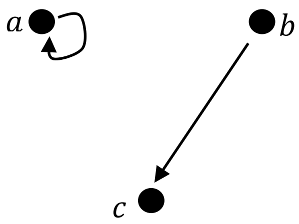
Answers:

Let $A = \{a, b, c\}$.

(a) $R = \{(a, b), (a, c)\}$.

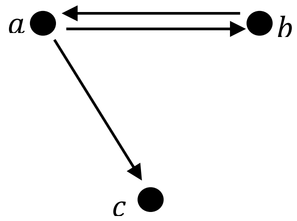


(b) $R = \{(a, a), (b, c)\}$



(c) No solution.
Every asymmetric relation is antisymmetric (see proof below).

(d) $R = \{(a, b), (b, a), (a, c)\}$



Proof: (Every asymmetric relation is antisymmetric.)

1. Take any binary relation R on a set A.
2. Suppose R is asymmetric.
   - 2.1. Then $\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x)$ by the definition of asymmetry.
   - 2.2. $\equiv \forall x, y \in A\ (x \not R\ y \vee y \not R\ x)$ by the implication law.
   - 2.3. $\Rightarrow \forall x, y \in A\ ((x \not R\ y \vee y \not R\ x) \vee x = y)$ by generalization.
   - 2.4. $\equiv \forall x, y \in A\ ({\sim}(x\ R\ y \wedge y\ R\ x) \vee x = y)$ by the De Morgan's law.
   - 2.5 $\equiv \forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y)$ by the implication law.
3. Line 2.5 is the definition of antisymmetry, hence R is antisymmetric.

We see that asymmetry property forces the antisymmetry property to be vacuously true.

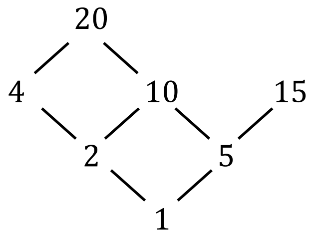
9. **Definitions.** Consider a partial order $\preccurlyeq$ on a set A and let $a, b \in A$.

- We say a, b are **comparable** iff $a \preccurlyeq b$ or $b \preccurlyeq a$.
- We say a, b are **compatible** iff there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

Consider the "divides" relation on $A = \{1, 2, 4, 5, 10, 15, 20\}$. List out the pairs of distinct elements in A that are (a) comparable; (b) compatible. Use the notation $\{x, y\}$ to represent the pair of elements x and y.

Answers:

Hasse diagram



(a) Comparable:

$\{1,2\}$, $\{1,5\}$, $\{1,4\}$, $\{1,10\}$, $\{1,15\}$, $\{1,20\}$, $\{2,4\}$, $\{2,10\}$, $\{2,20\}$, $\{5,10\}$, $\{5,15\}$, $\{5,20\}$, $\{4,20\}$ and $\{10,20\}$.

Not comparable: $\{2,5\}$, $\{2,15\}$, $\{5,4\}$, $\{4,10\}$, $\{4,15\}$, $\{10,15\}$ and $\{15,20\}$.

(b) Compatible:

$\{1,2\}$, $\{1,5\}$, $\{1,4\}$, $\{1,10\}$, $\{1,15\}$, $\{1,20\}$, $\{2,4\}$, $\{2,5\}$, $\{2,10\}$, $\{2,20\}$, $\{5,4\}$, $\{5,10\}$, $\{5,15\}$, $\{5,20\}$, $\{4,10\}$, $\{4,20\}$ and $\{10,20\}$.

Not compatible: $\{2,15\}$, $\{4,15\}$, $\{10,15\}$ and $\{15,20\}$.

10. [AY2022/23 Semester 2 Mid-term Test]
Let $\preccurlyeq$ be a partial order on a non-empty set A. A subset C of A is called a **chain** if and only if every pair of elements in C is comparable, that is, $\forall a, b \in C\ (a \preccurlyeq b \vee b \preccurlyeq a)$. A **maximal chain** is a chain M such that $t \notin M \Rightarrow M \cup \{t\}$ is not a chain. The length of a chain is one less than the number of elements in it.

(a) Let $A = \{a, b, c, d\}$ and $(\mathcal{P}(A), \subseteq)$ be a poset on $\mathcal{P}(A)$, where $\mathcal{P}(A)$ denotes the power set of A. Write out two maximal chains in $(\mathcal{P}(A), \subseteq)$.

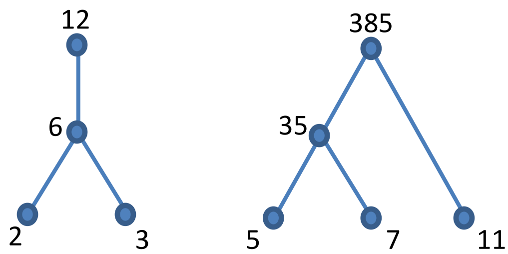
(b) Let $B = \{2,3,5,6,7,11,12,35,385\}$ and $(B, |)$ be a poset on B, where $|$ denotes the divides relation. Draw the Hasse diagram and write out two maximal chains of different lengths in $(B, |)$.

Answers:

(a) Many possible answers.

Two of them are $\{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,c,d\}\}$ and $\{\emptyset, \{b\}, \{b,d\}, \{b,c,d\}, \{a,b,c,d\}\}$.

(b) Hasse diagram:



Maximal chains of different lengths:

{2,6,12} and {11,385},
or {3,6,12} and {11,385},
or {5,35,385} and {11,385},
or {7,35,385} and {11,385}.

11. For each of the following statements, state whether it is true or false and justify your answer.

(a) In all partially ordered sets, any two comparable elements are compatible.

(b) In all partially ordered sets, any two compatible elements are comparable.

Let $\preccurlyeq$ be a partial order $\preccurlyeq$ on a set A and let $a, b \in A$.
- We say a, b are **comparable** iff $a \preccurlyeq b$ or $b \preccurlyeq a$.
- We say a, b are **compatible** iff there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

Answers:

(a) **True.** Proof:

1. Let $a, b \in A$ such that a and b are comparable.
2. Then either $a \preccurlyeq b$ or $b \preccurlyeq a$ by the definition of comparability.
3. Case 1: $a \preccurlyeq b$
   3.1. Let $c = b$.
   3.2. Then $a \preccurlyeq b = c$ by assumption and
   $b \preccurlyeq b = c$ by the reflexivity of $\preccurlyeq$.
   3.3. Hence a and b are compatible by the definition of compatibility.
4. Case 2: $b \preccurlyeq a$
   4.1. Let $c = a$.
   4.2. Then $b \preccurlyeq a = c$ by assumption and
   $a \preccurlyeq a = c$ by the reflexivity of $\preccurlyeq$.
   4.3. Hence a and b are compatible by the definition of compatibility.
5. In all cases, a and b are compatible.

Note that the 2 cases are symmetrical and hence very similar (just switch the roles of a and b). We may use WLOG (without loss of generality) to cover one of the cases. (Use WLOG with care! Make sure that the cases are indeed similar before you use WLOG.)

1. Let $a, b \in A$ such that a and b are comparable.
2. Then either $a \preccurlyeq b$ or $b \preccurlyeq a$ by the definition of comparability.
3. WLOG, let $a \preccurlyeq b$.
   3.1. Let $c = b$.
   3.2. Then $a \preccurlyeq b = c$ by assumption and
   $b \preccurlyeq b = c$ by the reflexivity of $\preccurlyeq$.
   3.3. Hence a and b are compatible by the definition of compatibility.
4. Hence a and b are compatible.

(b) **False.** Consider the poset $(\mathbb{Z}^+, |)$ where $|$ is the "divides" relation. 2 and 3 are compatible as $2 \mid 6$ and $3 \mid 6$. However, 2 and 3 are not comparable as $2 \nmid 3$ and $3 \nmid 2$.