CS1231S: Discrete Structures

Tutorial #5: Relations & Partial Orders

(Week 7: 30 September – 4 October 2024)

I. Discussion Questions

D1. Let R be a binary relation on a non-empty set A. If $R = \emptyset$, then is R reflexive? Symmetric? Transitive?

D2. Suppose a binary relation R on a non-empty set A is reflexive, transitive, symmetric and antisymmetric. What can you conclude about R? Explain.

D3. Asymmetry is defined in question 6 as follows: A binary relation R on a set A is **asymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x).$$

Are there binary relations that are both symmetric and asymmetric?

D4. Let R be a relation on A. Prove that the following is an alternative definition of **antisymmetry**:

$$\forall x, y \in A \Big((x \neq y) \Rightarrow \Big(\big((x, y) \in R\big) \Rightarrow \big((y, x) \notin R\big) \Big) \Big).$$

II. Tutorial Questions

1. Let S be the set of all strings over the alphabet $\mathcal{A} = \{s, u\}$, i.e. an element of S is a sequence of characters, each of which is either s or u. Examples of elements of S are: ε (the empty string), $s, u, sus, usssuu$, and $susussssu$.

   Define a relation R on S by the following: $\forall a, b \in S\ \big(aRb\ \Leftrightarrow len(a) \leq len(b)\big)$
   where $len(x)$ denotes the length of x, i.e. the number of characters in x.

   Is R a partial order? Prove or disprove it.

2. [AY2022/23 Semester 1 Midterm Test]
   Let $A = \{2,3,5,7,21,30,84,99\}$ and let $\preccurlyeq$ be a partial order on the set A defined by the "divides" relation, that is, $x \preccurlyeq y\ \Leftrightarrow x|y$. Which of the following statements are true?

   (i) The partial order has a linearization $\preccurlyeq^*$ such that $21 \preccurlyeq^* 7$.
   (ii) The partial order has a linearization $\preccurlyeq^*$ such that $3 \preccurlyeq^* 2$.
   (iii) The partial order has a linearization $\preccurlyeq^*$ such that $21 \preccurlyeq^* 5 \preccurlyeq^* 84$.
   (iv) The partial order has a linearization $\preccurlyeq^*$ such that $99 \preccurlyeq^* 84 \preccurlyeq^* 30$.

3. [AY2023/24 semester 1 Midterm Test]
   Let $A = \{11,12,13,14,15,16\}$. For each $x \in A$, define $F_x = \{k \in \mathbb{Z}^+ : k|x\}$, where $|$ is the "divides" relation. Define also a partial order $\preccurlyeq$ on A by setting for all $x, z \in A$:

$$x \preccurlyeq z\ \Leftrightarrow (F_x = F_z) \vee (|F_x| <\ |F_z|).$$

   What are the minimal, smallest, maximal, and largest elements of A with respect to $\preccurlyeq$?

4. Given the partial order $\preccurlyeq$ on A in question 3 above, one of the linearizations $\preccurlyeq^*$ of $\preccurlyeq$ is shown below:

$$11 \preccurlyeq^* 13 \preccurlyeq^* 14 \preccurlyeq^* 15 \preccurlyeq^* 16 \preccurlyeq^* 12$$

Write out all the other possible linearizations $\preccurlyeq^*$.

5. Let $\mathcal{P}(A)$ denote the power set of set A. Prove that the binary relation $\subseteq$ on $\mathcal{P}(A)$ is a partial order.

6. Let $B = \{0,1\}$ and define the binary relation R on $B \times B$ as follows:

$$\forall (a,b),(c,d) \in B \times B \left((a,b)\ R\ (c,d) \Leftrightarrow (a \le c) \wedge (b \le d)\right).$$

   (a) Prove that R is a partial order.
   (b) Draw the Hasse diagram for R.
   (c) Find the maximal, largest, minimal and smallest elements.
   (d) Is $(B \times B, R)$ well-ordered?

7. Let R be a binary relation on a non-empty set A. Let $x, y \in A$. Define a relation S on A by

$$x\ S\ y \Leftrightarrow (x = y) \vee (x\ R\ y) \text{ for all } x, y \in A.$$

   Show that:
   (a) S is reflexive;
   (b) $R \subseteq S$; and
   (c) if S' is another reflexive relation on A and $R \subseteq S'$, then $S \subseteq S'$.

   What is this relation S called? (Hint: Refer to Transitive Closure in Lecture 6).

8. Let R be a binary relation on a set A.
   We have defined antisymmetry in class: R is **antisymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y).$$

   We define asymmetry here. R is **asymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x).$$

   (a) Find a binary relation on A that is both asymmetric and antisymmetric.
   (b) Find a binary relation on A that is not asymmetric but antisymmetric.
   (c) Find a binary relation on A that is asymmetric but not antisymmetric.
   (d) Find a binary relation on A that is neither asymmetric nor antisymmetric.

9. **Definitions.** Consider a partial order $\preccurlyeq$ on a set A and let $a, b \in A$.

   - We say a, b are **comparable** iff $a \preccurlyeq b$ or $b \preccurlyeq a$.
   - We say a, b are **compatible** iff there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

   Consider the "divides" relation on $A = \{1, 2, 4, 5, 10, 15, 20\}$. List out the pairs of distinct elements in A that are (a) comparable; (b) compatible. Use the notation $\{x, y\}$ to represent the pair of elements x and y.

10. [AY2022/23 Semester 2 Mid-term Test]
    Let $\preccurlyeq$ be a partial order on a non-empty set A. A subset C of A is called a **chain** if and only if every pair of elements in C is comparable, that is, $\forall a, b \in C\ (a \preccurlyeq b \vee b \preccurlyeq a)$. A **maximal chain** is a chain M such that $t \notin M \Rightarrow M \cup \{t\}$ is not a chain. The length of a chain is one less than the number of elements in it.

    (a) Let $A = \{a, b, c, d\}$ and $(\mathcal{P}(A), \subseteq)$ be a poset on $\mathcal{P}(A)$, where $\mathcal{P}(A)$ denotes the power set of A. Write out two maximal chains in $(\mathcal{P}(A), \subseteq)$.

    (b) Let $B = \{2,3,5,6,7,11,12,35,385\}$ and $(B, |)$ be a poset on B, where $|$ denotes the divides relation. Draw the Hasse diagram and write out two maximal chains of different lengths in $(B, |)$.

11. For each of the following statements, state whether it is true or false and justify your answer.

    (a) In all partially ordered sets, any two comparable elements are compatible.

    (b) In all partially ordered sets, any two compatible elements are comparable.