CS1231S: Discrete Structures

Tutorial #6: Functions

(Week 8: 7 – 11 October 2024)

I. Discussion Questions

D1. Which of the following is a function? If it is not a function, explain.

(a) Define $f: \mathbb{Z} \to \mathbb{Z}$ by $\forall z \in \mathbb{Z}, f(z) = \begin{cases} 1, & \text{if } 2 \mid z, \\ 2, & \text{if } 3 \mid z. \end{cases}$

(b) Define $f: \mathbb{Z} \to \mathbb{Z}$ by $\forall z \in \mathbb{Z}, f(z) = \begin{cases} 1, & \text{if } 2 \mid z, \\ 2, & \text{if } 2 \nmid z. \end{cases}$

(c) Define $f: \mathbb{R} \to \mathbb{Z}$ by $\forall x \in \mathbb{R}, f(x) = 2x$.

(d) Define $f: \mathbb{Z} \to \mathbb{R}$ by $\forall x \in \mathbb{Z}, f(x) = 2x$.

D2. Let function $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ be defined by setting, $\forall x. y \in \mathbb{Z}, f(x, y) = \frac{x+y}{3}$.
Find three distinct pre-images of 2.

D3. Definitions: Given any real number x,

(1) the **floor** of x, denoted $\lfloor x \rfloor$, is the unique integer n such that $n \le x < n + 1$;
(2) the **ceiling** of x, denoted $\lceil x \rceil$, is the unique integer n such that $n - 1 < x \le n$.

Let $f, g: \mathbb{Q} \to \mathbb{Q}$ be defined by setting, for each $x \in \mathbb{Q}$,

$$f(x) = \lfloor x \rfloor + 1 \quad \text{and} \quad g(x) = \lceil x \rceil.$$

What is the range of f? What is the range of g? Is $f = g$? Why?

D4. To prove that a composition of two surjections is a surjection, Aiken wrote:

1. Suppose $f: X \to Y$ and $g: Y \to Z$ are surjections.
2. Then $\forall y \in Y\ \exists x \in X$ such that $f(x) = y$ as f is surjective,
3. and $\forall z \in Z\ \exists y \in Y$ such that $g(y) = z$ as g is surjective.
4. So $(g \circ f)(x) = g\big(f(x)\big) = g(y) = z$.
5. Hence $g \circ f$ is a surjection.

Explain the mistakes in this "proof".

II. Tutorial Questions

1. Define the following relations on $\mathbb{N}$:

   $\forall x, y \in \mathbb{N}\ (x\ R_1\ y\ \Leftrightarrow x^2 = y^2)$;

   $\forall x, y \in \mathbb{N}\ (x\ R_2\ y\ \Leftrightarrow y \mid x)$;

   $\forall x, y \in \mathbb{N}\ (x\ R_3\ y\ \Leftrightarrow y = x + 1)$.

   Are the relations R_1, R_2 and R_3 functions? Prove or disprove.

2. Let $A = \{a, b\}$ and S be the set of all strings over A. (See Lecture 7 section 7.1.2 for the definition of string.)

   Define a concatenate-by-a-on-the-left function $C: S \to S$ by $C(s) = as$ for all $s \in S$.

   (a) Is C an injection? Prove or give a counterexample.

   (b) Is C a surjection? Prove or give a counterexample.

3. Let $A = \{s, u\}$. Define a function $len: A^* \to \mathbb{Z}_{\geq 0}$ by setting $len(\sigma)$ to be the length of σ for each $\sigma \in A^*$.

   (a) What is $len(suu)$?

   (b) What is $len(\{\varepsilon, ss, uu, ssss\})$?

   (c) What is $len^{-1}(\{3\})$?

   (d) Does len^{-1} exist? Explain your answer.

4. Given any two bijections $f: A \to B$ and $g: B \to C$, prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

5. Which of the functions defined in the following are injective? Which are surjective? Prove that your answers are correct. If a function defined below is both injective and surjective, then find a formula for the inverse of the function. Here we denote by $Bool$ the set {**true**, **false**}.

   (a) $f: \mathbb{Q} \to \mathbb{Q}$;
   $x \mapsto 12x + 31$.

   (b) $g: Bool^2 \to Bool$;
   $(p, q) \mapsto p \wedge {\sim}q$.

   (c) $h: Bool^2 \to Bool^2$;
   $(p, q) \mapsto (p \wedge q, p \vee q)$.

   (d) $k: \mathbb{Z} \to \mathbb{Z}$;
   $x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$.

6. [AY2022/23 Semester 2 Exam Questions]
The following definitions are given.

> Given a function $f: A \to B$, we say that
> - $g: B \to A$ is a **left inverse** of f if and only if $g\big(f(a)\big) = a$ for all $a \in A$.
> - $h: B \to A$ is a **right inverse** of f if and only if $f\big(h(b)\big) = b$ for all $b \in B$.

You do not need to provide proofs for the following parts.

(a) Which of the 4 functions given in question 5 have a left inverse?

(b) Which of the 4 functions given in question 5 have a right inverse?

(c) Which of the following statements are true?

  (i) If a function has a left inverse, then it has a right inverse.

  (ii) If a function has a right inverse, then it has a left inverse.

7. We have shown in Theorem 7.3.3 that if $f: X \to Y$ and $g: Y \to Z$ are both injective, then $g \circ f$ is injective.

Now, let $f: B \to C$. Suppose we have a function g with domain C such that $g \circ f$ is injective. Show that f is injective.

8. Let $A = \{1,2,3\}$. The ***order*** of a bijection $f: A \to A$ is defined to be the smallest $n \in \mathbb{Z}^+$ such that

$$\underbrace{f \circ f \circ \cdots \circ f}_{n\text{-many } f\text{'s}} = id_A.$$

Define functions $g, h: A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

Find the orders of $g, h, g \circ h$, and $h \circ g$.

9. Let $f: A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$. Justify your answers for the following:

(a) Is it always the case that $X \subseteq \boldsymbol{f}^{-1}\big(\boldsymbol{f}(X)\big)$? Is it always the case that $\boldsymbol{f}^{-1}\big(\boldsymbol{f}(X)\big) \subseteq X$?

(b) Is it always the case that $Y \subseteq \boldsymbol{f}\left(\boldsymbol{f}^{-1}(Y)\right)$? Is it always the case that $\boldsymbol{f}\left(\boldsymbol{f}^{-1}(Y)\right) \subseteq Y$?

Note that $\boldsymbol{f}(X)$ is the **(setwise) image** of X, and $\boldsymbol{f}^{-1}(Y)$ the **(setwise) preimage** of Y under f, where $X \subseteq A$ and $Y \subseteq B$. Without the colored font to disambiguate the two kinds of functions, it should also be clear what $f(U)$ denotes, depending on whether $U \in A$ or $U \in P(A)$.

10. Consider the equivalence relation $\sim$ on $\mathbb{Q}$ defined by setting, for all $x, y \in \mathbb{Q}$,

$$x \sim y \Leftrightarrow x - y \in \mathbb{Z}.$$

Define addition and multiplication on $\mathbb{Q}/\sim$ as follows: whenever $[x], [y] \in \mathbb{Q}/\sim$,

$$[x] + [y] = [x + y] \quad \text{and} \quad [x] \cdot [y] = [x \cdot y].$$

(a) Is $+$ well defined on $\mathbb{Q}/\sim$?

(b) Is $\cdot$ well defined on $\mathbb{Q}/\sim$?

Prove that your answers are correct.

11. Consider the addition function + under $\mathbb{Q}$.

Consider a new function $+$ you constructed in Q10 for addition on $\mathbb{Q}/\sim$ as follows:

whenever $[x], [y] \in \mathbb{Q}/\sim$, we have $[x] + [y] = [x + y]$.

The type signature for function + under $\mathbb{Q}$ would be:

$+ : (\mathbb{Q}, \mathbb{Q}) \rightarrow \mathbb{Q}$

What is the type signature of the new function $+$ which you just constructed via the above equality relation?