CS1231S: Discrete Structures

Tutorial #7: Mathematical Induction and Recursion

(Week 9: 14 – 18 October 2024)

Answers

In writing Mathematical Induction proofs, please follow the format we use in class.

1. Many common mistakes are spotted in Mathematical Induction proofs. For each of the following, point out the mistake. (If you follow the format we introduced in class, it will help you avoid making some of these mistakes.)

   Consider this problem: Prove by mathematical induction that for all $n \in \mathbb{Z}^+$,

   $$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

   Define the statement $P(n)$ to be "$1 + 2 + \cdots n = n(n+1)/2$".

   (a) Inductive hypothesis: Assume that $P(k)$ is true.
   Inductive step: $\sum_{i=1}^{k+1} i = (k+1) + \sum_{i=1}^{k} i = (k+1) + P(k) = \cdots$

   (b) Inductive step:
   1. $1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$
   2. Hence $\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$ (by the inductive hypothesis)
   3. Hence $\frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$ (by basic algebra)
   4. Hence $\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$.
   5. Since (4) is a tautology, (1) must be true.

   (c) Inductive hypothesis: Assume that $P(k)$ is true for all $k \in \mathbb{Z}^+$.

   Answers:

   (a) Logical propositions/statements like $P(k)$ are <u>not</u> values and cannot be added to numbers!

   (b) (Proof going the wrong way.) Make sure you use $P(k)$ to prove $P(k+1)$ and not the other way round. The proof here starts off with $P(k+1)$ and ends using $P(k)$ to prove an identity, which does not prove anything. Make sure you do not assume $P(k+1)$!

   (c) (Assuming too much.) Note that "Assume that $P(k)$ is true for all $k \in \mathbb{Z}^+$" is the same statement as "Assume that $P(n)$ is true for all $n \in \mathbb{Z}^+$" which is what you are supposed to prove. So, there is nothing left to prove! Make sure you don't assume everything in the inductive hypothesis.

   Note: This question is based on
   http://www.cs.cmu.edu/~arielpro/15251f17/notes/induction-pitfalls.pdf

2. Prove by induction that for all $n \in \mathbb{Z}^+$,

$$\mathbf{1^2 + 2^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1).}$$

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n) \equiv 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.
2. (Basis step) $P(1)$ is true because $1^2 = 1 = \frac{1}{6} \times 1 \times (1+1) \times (2 \times 1 + 1)$.
3. (Inductive step)
   3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e. $1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$
   3.2. Then $1^2 + 2^2 + \cdots + k^2 + (k+1)^2$
   3.3. $= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$ by the inductive hypothesis
   3.4. $= \frac{1}{6}(k+1)\big(k(2k+1) + 6(k+1)\big)$
   3.5. $= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$
   3.6. $= \frac{1}{6}(k+1)(k+2)(2k+3)$
   3.7. $= \frac{1}{6}(k+1)\big((k+1)+1\big)(2(k+1)+1)$ by basic algebra
   3.8. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+\ P(n)$ is true by MI.

3. Let $x \in \mathbb{R}_{\geq -1}$. Prove by induction that $\mathbf{1 + nx \leq (1+x)^n}$ for all $n \in \mathbb{Z}^+$.

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n) \equiv (1 + nx \leq (1+x)^n)$.
2. (Basis step) $P(1)$ is true because $1 + 1x = 1 + x = (1+x)^1$.
3. (Inductive step)
   3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e. $1 + kx \leq (1+x)^k$.
   3.2. Then $(1+x)^{k+1}$
   3.3. $= (1+x)^k(1+x)$
   3.4. $\geq (1+kx)(1+x)$ by the inductive hypothesis
   3.5. $= 1 + (k+1)x + kx^2$
   3.6. $\geq 1 + (k+1)x$ as $k \geq 1$ and $x^2 \geq 0$, so $kx^2 \geq 0$
   3.7. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+\ P(n)$ is true by MI.

4. Let a be an odd integer. Prove by induction that $\mathbf{2^{n+2} \mid (a^{2^n} - 1)}$ for all $n \in \mathbb{Z}^+$.
Here you may use without proof the fact that the product of any two consecutive integers is even. (Prove that as your own exercise – it's very simple.)
(Note that $a^{b^c} = a^{(b^c)}$ by convention.)

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n) \equiv 2^{n+2} \mid a^{2^n} - 1$.
2. (Basis step)
   2.1. $a = 2p + 1$ for some integer p by the definition of odd integers.
   2.2. Then $a^{2^1} - 1 = a^2 - 1 = (a-1)(a+1) = (2p+1-1)(2p+1+1) = 4p(p+1)$.
   2.3. Now $p(p+1)$ is even (given by the question), so $p(p+1) = 2m$ for some integer m by the definition of even integers.
   2.4. Hence, $a^{2^1} - 1 = 4(2m) = 8m = 2^3 m$.
   2.5. So $2^{1+2} \mid a^{2^1} - 1$ and hence $P(1)$ is true.
3. (Inductive step)
   3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e., $2^{k+2} \mid a^{2^k} - 1$.
   3.2. So $a^{2^k} - 1 = 2^{k+2}m$ for some integer m by the definition of divisibility.
   3.3. Then $a^{2^{k+1}} - 1 = a^{2^k \times 2} - 1$
   3.4. $= \left(a^{2^k}\right)^2 - 1$
   3.5. $= (a^{2^k} - 1)(a^{2^k} + 1)$
   3.6. $= (2^{k+2}m)\left((2^{k+2}m + 1) + 1\right)$ by line 3.2
   3.7. $= 2^{k+3}m(2^{k+1}m + 1)$
   3.8. Thus $2^{k+3}m \mid a^{2^{k+1}} - 1$ and so $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+\ P(n)$ is true by MI.

5. Prove by induction that

$$\forall \boldsymbol{n} \in \mathbb{Z}_{\geq 8}\ \exists \boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}\ (\boldsymbol{n} = \boldsymbol{3x} + \boldsymbol{5y}).$$

(In other words, any integer-valued transaction of at least $8 can be carried out using only $3 and $5 notes.)

Answer:

1. For each $n \in \mathbb{Z}_{\geq 8}$, let $P(n) \equiv \exists x, y \in \mathbb{N}\ (n = 3x + 5y)$.
2. (Basis step) $P(8)$ is true as $8 = 3(1) + 5(1)$.
3. (Inductive step)
    - 3.1. Let $k \in \mathbb{Z}_{\geq 8}$ such that $P(k)$ is true.
    - 3.2. Find $x, y \in \mathbb{N}$ such that $k = 3x + 5y$.
    - 3.3. Case 1: $y > 0$.
        - 3.3.1. Then $k + 1 = (3x + 5y) + 1$ by the choice of x, y.
        - 3.3.2. $= 3(x + 2) + 5(y - 1)$.
        - 3.3.3. $y - 1 \in \mathbb{N}$ as $y > 0$.
        - 3.3.4. As $x + 2 \in \mathbb{N}$ and $y - 1 \in \mathbb{N}$, so $P(k + 1)$ is true.
    - 3.4. Case 2: $y = 0$.
        - 3.4.1. Then $k = 3x + 5(0) = 3x$.
        - 3.4.2. $\therefore x = k/3 \geq 8/3$ as $k \geq 8$.
        - 3.4.3. $\therefore x \geq 3$ as $x \in \mathbb{N}$.
        - 3.4.4. Thus $k + 1 = 3x + 1 = 3(x - 3) + 5(2)$.
        - 3.4.5. As $x - 3 \in \mathbb{N}$ and $2 \in \mathbb{N}$, So $P(k + 1)$ is true.
    - 3.5. Hence $P(k + 1)$ is true for all cases.
4. Therefore, $\forall n \in \mathbb{N}\ P(n)$ is true by MI.

Alternative answer:

1. For each $n \in \mathbb{Z}_{\geq 8}$, let $P(n) \equiv \exists x, y \in \mathbb{N}\ (n = 3x + 5y)$.
2. (Basis step)
    - 2.1. $P(8)$ is true because $8 = 3(1) + 5(1)$.
    - 2.2. $P(9)$ is true because $9 = 3(3) + 5(0)$.
    - 2.3. $P(10)$ is true because $10 = 3(0) + 5(2)$.
3. (Inductive step)
    - 3.1. Let $k \in \mathbb{Z}_{\geq 8}$ such that $P(8), P(9), \cdots, P(k + 2)$ are true.
    - 3.2. Apply $P(k)$ to find $x, y \in \mathbb{N}$ such that $k = 3x + 5y$.
    - 3.3. Then $k + 3 = (3x + 5y) + 3$ by the choice of x, y.
    - 3.4. $= 3(x + 1) + 5y$ where $x + 1, y \in \mathbb{N}$.
    - 3.5. Hence $P(k + 3)$ is true.
4. Therefore, $\forall n \in \mathbb{N}\ P(n)$ is true by Strong MI.

6. Prove by induction that every positive integer can be written as a sum of *distinct* non-negative integer powers of 2, i.e.,

$$\forall \boldsymbol{n} \in \mathbb{Z}^+ \ \exists \boldsymbol{l} \in \mathbb{Z}^+ \ \exists \boldsymbol{i_1}, \boldsymbol{i_2}, \cdots, \boldsymbol{i_l} \in \mathbb{N} \ (\boldsymbol{i_1} < \boldsymbol{i_2} < \cdots < \boldsymbol{i_l} \ \wedge \boldsymbol{n} = \mathbf{2}^{\boldsymbol{i_1}} + \mathbf{2}^{\boldsymbol{i_2}} + \cdots + \mathbf{2}^{\boldsymbol{i_l}}).$$

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n)$ be the proposition
   "$\exists l \in \mathbb{Z}^+ \ \exists i_1, i_2, \cdots, i_l \in \mathbb{N} \ (i_1 < i_2 < \cdots < i_l \ \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l})$"
2. (Basis step) $P(1)$ is true as $1 = 2^0$.
3. (Inductive step)
   3.1. Let $k \in \mathbb{Z}^+$ such that $P(1), P(2), \cdots, P(k)$ are true.
   3.2. Find $m \in \mathbb{Z}$ such that $k + 1 = 2m$ or $k + 1 = 2m + 1$. (This is possible because $k + 1$ is either even or odd, by lecture #1 assumption 1.)
   3.3. Note that $2m \leq k + 1$ as $k + 1 = 2m$ or $k + 1 = 2m + 1$;
   3.4. $\leq k + k$ as $k \geq 1$;
   3.5. $= 2k$.
   3.6. So $m \leq k$.
   3.7. Also, $2m + 1 \geq k + 1$ as $k + 1 = 2m$ or $k + 1 = 2m + 1$;
   3.8. So $2m \geq k \geq 1$ or $m \geq \frac{1}{2}$ or $m \geq 1$ as $m \in \mathbb{Z}$ and 1 is the smallest integer $\geq \frac{1}{2}$.
   3.9. By lines 3.6 and 3.8, $1 \leq m \leq k$, and so $P(m)$ is true by the inductive hypothesis.
   3.10. Apply $P(m)$ to find $l \in \mathbb{Z}^+$ and $i_1, i_2, \cdots, i_l \in \mathbb{N}$ such that
   $$i_1 < i_2 < \cdots < i_l \quad \text{and} \quad m = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}.$$
   3.11. Case 1: $k + 1 = 2m$.
      3.11.1. Then $k + 1 = 2(2^{i_1} + 2^{i_2} + \cdots + 2^{i_l})$
      3.11.2. $= 2^{i_1+1} + 2^{i_2+1} + \cdots + 2^{i_l+1}$.
      3.11.3. Also, $i_1 + 1 < i_2 + 1 < \cdots < i_l + 1$ as $i_1 < i_2 < \cdots < i_l$.
      3.11.4. So $P(k + 1)$ is true.
   3.12. Case 1: $k + 1 = 2m + 1$.
      3.12.1. Then $k + 1 = 2(2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}) + 1$
      3.12.2. $= 2^0 + 2^{i_1+1} + 2^{i_2+1} + \cdots + 2^{i_l+1}$.
      3.12.3. Also, $0 < i_1 + 1 < i_2 + 1 < \cdots < i_l + 1$ as $0 \leq i_1 < i_2 < \cdots < i_l$.
      3.12.4. So $P(k + 1)$ is true.
   3.13. Hence $P(k + 1)$ is true for all cases.
4. Therefore, $\forall n \in \mathbb{Z}^+ \ P(n)$ is true by Strong MI.

Alternative answer (not for the faint-hearted):

1. For each $n \in \mathbb{Z}^+$, let $P(n)$ be the proposition

   $$\text{“}\exists l \in \mathbb{Z}^+\ \exists i_1, i_2, \cdots, i_l \in \mathbb{N}\ (i_1 < i_2 < \cdots < i_l\ \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l})\text{”}$$

2. (Basis step) $P(1)$ is true as $1 = 2^0$.
3. (Inductive step)

   3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true.

   3.2. Apply this assumption to obtain $l \in \mathbb{Z}^+$ and $i_1, i_2, \cdots, i_l \in \mathbb{N}$ such that

   $$i_1 < i_2 < \cdots < i_l \quad \text{and} \quad k = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}.$$

   3.3. Since $k < 2^k$, we know k is an element of the set $\mathbb{N} \setminus \{i_1, i_2, \cdots, i_l\}$.

   3.4. By the Well-Ordering Principle, this set has a minimum, say m.

   3.5. Find $j \in \{0,1, \cdots, l\}$ such that $i_1 < i_2 < \cdots < i_j < m < i_{j+1} < \cdots < i_l$.

   3.6. The minimality of m tells us $0,1,2, \cdots, m-1 \in \{i_1, i_2, \cdots, i_l\}$.

   3.7. Thus $0,1,2, \cdots, m-1 \in \{i_1, i_2, \cdots, i_j\}$ by the choice of j.

   3.8. The choice of j also tell us $i_1, i_2, \cdots, i_j \in \{0,1,2, \cdots, m-1\}$.

   3.9. From these, we deduce that $\{i_1, i_2, \cdots, i_j\} = \{0,1,2, \cdots, m-1\}$.

   3.10. Now, $k + 1 = 1 + 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}$ by line 3.2;

   3.11. $= 2^0 + 2^0 + 2^1 + 2^2 + \cdots + 2^{m-1} + 2^{i_{j+1}} + \cdots + 2^{i_l}$ by line 3.9;

   3.12. $= 2^1 + 2^1 + 2^2 + \cdots + 2^{m-1} + 2^{i_{j+1}} + \cdots + 2^{i_l}$

   3.13. $= 2^2 + 2^2 + \cdots + 2^{m-1} + 2^{i_{j+1}} + \cdots + 2^{i_l}$

   3.14. $= \cdots = 2^{m-1} + 2^{m-1} + 2^{i_{j+1}} + \cdots + 2^{i_l}$

   3.15. $= 2^m + 2^{i_{j+1}} + \cdots + 2^{i_l}$

   3.16. So $P(k+1)$ is true.

4. Therefore, $\forall n \in \mathbb{Z}^+\ P(n)$ is true by MI.

7. Let $a_0, a_1, a_2 \cdots$ be the sequence satisfying

$$a_0 = 0, \quad a_1 = 2, \quad a_2 = 7, \quad \text{and} \quad a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

for all $n \in \mathbb{N}$. Prove by induction that $\boldsymbol{a_n < 3^n}$ for all $n \in \mathbb{N}$.

Answer:

1. For each $n \in \mathbb{N}$, let $P(n) \equiv a_n < 3^n$.
2. (Basis step)

   $P(0), P(1), P(2)$ are true as $a_0 = 0 < 1 = 3^0$, $a_1 = 2 < 3 = 3^1$, and $a_2 = 7 < 9 = 3^2$.

3. (Inductive step)

   3.1. Let $k \in \mathbb{N}$ such that $P(0), P(1), \cdots, P(k+2)$ are true.

   3.2. $P(k), P(k+1), P(k+2)$ are true means $a_k < 3^k, a_{k+1} < 3^{k+1}, a_{k+2} < 3^{k+2}$.

   3.3. $a_{k+3} = a_{k+2} + a_{k+1} + a_k$ by the definition of a_{k+3};

   3.4. $< 3^{k+2} + 3^{k+1} + 3^k$ by the inductive hypothesis;

   3.5. $< 3^{k+2} + 3^{k+2} + 3^{k+2}$

   3.6. $= 3(3^{k+2}) = 3^{k+3}$.

   3.7. Thus $P(k+3)$ is true.

4. Therefore, $\forall n \in \mathbb{N}\ P(n)$ is true by Strong MI.

8. [Adapted from AY2023/24 Semester 1 exam]
Consider the Fibonacci function:

$$F(0) = 0; \qquad F(1) = 1; \qquad F(n+1) = F(n) + F(n-1), n \geq 1.$$

One interesting property of this function can be expressed as follows:

$$P(a,b) \equiv F(a+b) = \big(F(a+1) \times F(b) + F(a) \times F(b-1)\big),\ \forall a \geq 0, b \geq 1.$$

In your proofs for the parts below, you may use "basic algebra" for identity/commutative/associative laws of addition and multiplication, but if the distributive law is used, you must state it explicitly.

(a) There are a few induction schemes possible to solve part (b). For this question, use the following basis steps: $P(0,b)$ and $P(1,b)$, for all $b \geq 1$. Prove these basis steps.

(b) Let $b \in \mathbb{N}$. Prove that

$$P(n-1,b) \land P(n,b) \to P(n+1,b) \text{ for all } n \in \mathbb{Z}^+.$$

Answers:

(a) $P(0,b)$:

1. $F(0+b) = F(b)$
2. $= \big(1F(b) + 0F(b-1)\big)$
3. $= \big(F(1)F(b) + F(0)F(b-1)\big)$ by substituting $F(1) = 1$ and $F(0) = 0$
4. $= \big(F(0+1)F(b) + F(0)F(b-1)\big)$
5. Therefore $P(0,b)$ is true.

$P(1,b)$:

1. $F(1+b) = F(b+1)$
2. $= \big(1F(b) + 1F(b-1)\big)$ by the definition of Fibonacci
3. $= \big(F(2)F(b) + F(1)F(b-1)\big)$ by substituting $F(2) = 1$ and $F(1) = 1$
4. $= \big(F(1+1)F(b) + F(1)F(b-1)\big)$
5. Therefore $P(1,b)$ is true.

(b)

1. (Basis step) $P(0,b)$ and $P(1,b)$ proved in part (a).
2. (Inductive step)
   - 2.1. Let $b \in \mathbb{N}$. Let $k \in \mathbb{Z}^+$ such that $P(k-1,b) \land P(k,b)$.
   - 2.2. $F(k+1+b) = F(k+b+1)$ by basic algebra
   - 2.3. $= F(k+b) + F(k+b-1)$ by the definition of F
   - 2.4. $= \big(F(k+1)F(b) + F(k)F(b-1)\big) + F(k+b-1)$ by hypothesis $P(k,b)$
   - 2.5. $= (F(k+1)F(b) + F(k)F(b-1)) + \big(F(k)F(b) + F(k-1)F(b-1)\big)$ by hypothesis $P(k-1,b)$
   - 2.6. $= \big(F(k+1)F(b) + F(k)F(b)\big) + \big(F(k)F(b-1) + F(k-1)F(b-1)\big)$ by basic algebra
   - 2.7. $= \big(F(k+1) + F(k)\big)F(b) + \big(F(k) + F(k-1)\big)F(b-1)$ by distributive law x2
   - 2.8. $= F(k+2)F(b) + F(k+1)F(b-1)$ by the definition of F
   - 2.9. Hence, $P(k+1,b)$ is true. by the definition of P
3. Therefore, $P(n-1,b) \land P(n,b) \to P(n+1,b)$ for all $n \in \mathbb{Z}^+$.

9. The set H of Hamming numbers is recursively defined as follows.

(1) $1 \in H$. (base clause)

(2) If $n \in H$, then $2n \in H$ and $3n \in H$ and $5n \in H$ (recursion clause)

(3) Membership for H can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

A proof rule using regular induction for this set of Hamming numbers can be written as:

$$P(1)$$
$$\forall n \in H\ \big(P(n) \Rightarrow P(2n) \wedge P(3n) \wedge P(5n)\big)$$
$$\therefore\ \forall n \in H\ P(n)$$

Use this 1PI proof rule/axiom, to show that Hamming numbers have a canonical representation.

A set has a **canonical representation** if there is a unique way for representing each distinct member of the set. For example, the Hamming number built by the following three ways

$$2 \times \big(3 \times (3 \times 1)\big) \text{ or } 3 \times \big(2 \times (3 \times 1)\big) \text{ or } 3 \times \big(3 \times (2 \times 1)\big)$$

are equivalent to each other. Fortunately, we can represent Hamming numbers from this equivalent class of numbers in a canonical/unique manner by writing $2^1 3^2 5^0$.

We can prove that a canonical representation exists for H by proving the following property:

$$P(n) \equiv \exists{\sim} i\ \exists!\, j\ \exists!\, k\ \big((i, j, k \geq 0) \wedge n = 2^i 3^j 5^k\big)$$

Use 1PI to prove that this canonical representation exists for H.

Answer:

Structural induction over H. To prove that $\forall n \in H\ P(n)$ is true, where each $P(n)$ is as defined above:

(basis step) show that $P(1)$ is true; and

(inductive step) show that

$$\forall n \in H\ \big(P(n) \Rightarrow P(2n) \wedge P(3n) \wedge P(5n)\big)$$

is true.

Proof:

1. For each $n \in H$, prove $P(n)$.
2. (Basis step)
   - 2.1. $P(1)$ holds since 1 can be represented uniquely as $2^0 3^0 5^0$. That is
   $$\exists!\, i\ \exists!\, j\ \exists!\, k\ \big((i, j, k \geq 0) \wedge (1 = 2^i 3^j 5^k) \wedge (i = j = k = 0)\big)$$
   - 2.2. So $P(1)$ is true. (by definition of P)
3. (Inductive step)
   - 3.1. Let $n \in H$ such that $P(n)$ is true, i.e.,
   $$\exists!\, i\ \exists!\, j\ \exists!\, k\ \big((i, j, k \geq 0) \wedge (n = 2^i 3^j 5^k)\big)$$
   - 3.2. We can prove $P(2n)$
     - 3.2.1. $\exists!\, i\ \exists!\, j\ \exists!\, k\ \big((i, j, k \geq 0) \wedge (n = 2^i 3^j 5^k)\big)$ (from premise)
     - 3.2.2. $\exists!\, i\ \exists!\, j\ \exists!\, k\ \big((i, j, k \geq 0) \wedge (2n = 2^{i+1} 3^j 5^k)\big)$ (from basic algebra)
     - 3.2.3. $\exists i'\ \exists!\, i\ \exists!\, j\ \exists!\, k\ \big((i, j, k \geq 0) \wedge (2n = 2^{i'} 3^j 5^k) \wedge (i' = i + 1)\big)$ (adding $\exists i'$, $i' = i + 1$ that is equivalent to $true$)
     - 3.2.4. $\exists!\, i'\ \exists!\, j\ \exists!\, k\ \big((i', j, k \geq 0) \wedge (2n = 2^{i'} 3^j 5^k)\big)$ (unique i' due to bijective +1, unique i)
     - 3.2.5. Thus $P(2n)$ is true (by definition of P)
   - 3.3. We can prove $P(3n)$ (WLOG similar to 3.2)
   - 3.4. We can prove $P(5n)$ (WLOG similar to 3.2 and 3.3)
   - 3.5. Hence $P(2n)$ *and* $P(3n)$ *and* $P(5n)$ are all true in all cases.
4. It follows that $\forall n \in H, P(n)$ is true by structural induction over H.

10. Define a set S recursively as follows.

(1) $2 \in S$. (base clause)

(2) If $x \in S$, then $3x \in S$ and $x^2 \in S$. (recursion clause)

(3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the numbers $0, 6, 15, 16, 36$ are in S? Which are not?

Answer:

Structural induction over S. To prove that $\forall n \in S\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(basis step) show that $P(2)$ is true; and

(inductive step) show that $\forall x \in S\ \big(P(x) \Rightarrow P(3x) \wedge P(x^2)\big)$ is true.

- We know $0 \notin S$ because all $n \in S$ satisfy $n \geq 2$, as one can show by structural induction over S as follows.

  1. For each $n \in S$, let $P(n) \equiv n \geq 2$.
  2. (Basis step) $P(2)$ is true because $2 \geq 2$.
  3. (Inductive step)
     - 3.1. Let $x \in S$ such that $P(x)$ is true, i.e., that $x \geq 2$.
     - 3.2. Then $3x \geq 3 \times 2 = 6 \geq 2$ and $x^2 \geq 2^2 = 4 \geq 2$.
     - 3.3. So $P(3x)$ and $P(x^2)$ are both true.
  4. Hence $\forall n \in S\ P(n)$ is true by structural induction over S.

- $2 \in S$ by the base clause.
  $\therefore$ $6 \in S$ by the recursion clause with $n = 2$ and the previous line.
  $\therefore$ $36 \in S$ by the recursion clause with $n = 6$ and the previous line.

- $2 \in S$ by the base clause.
  $\therefore$ $4 \in S$ by the recursion clause with $n = 2$ and the previous line.
  $\therefore$ $16 \in S$ by the recursion clause with $n = 4$ and the previous line.

- We know $15 \notin S$ because no $n \in S$ is odd, as one can show by structural induction over S.

11. Let $A = \{1,2,3,4,5\}$ and $B = \{1,3,5,7,9\}$. Define a set S recursively as follows.

(1) $A, B \in S$. (base clause)

(2) If $X, Y \in S$, then $X \cap Y \in S$ and $X \cup Y \in S$ and $X \setminus Y \in S$ (recursion clause)

(3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

For each of the following sets, determine whether it is in S, and use one sentence to explain your answer.

(a) $C = \{2,4,7,9\}$.

(b) $D = \{2,3,4,5\}$.

Answer:

Structural induction over S. To prove that $\forall X \in S\ P(X)$ is true, where each $P(X)$ is a proposition, it suffices to:

(basis step) show that $P(A)$ and $P(B)$ are true; and

(inductive step) show that

$$\forall X, Y \in S\ \big(P(X) \wedge P(Y) \Rightarrow P(X \cap Y) \wedge P(X \cup Y) \wedge P(X \setminus Y)\big)$$

is true.

(a) $A \setminus B = \{2,4\}$ and $B \setminus A = \{7,9\}$.
$C \in S$ because $C = \{2,4,7,9\} = (A \setminus B) \cup (B \setminus A)$.

(b) $D \notin S$ because $1 \notin D$ and $3 \in D$, but one can show by structural induction that

$$\forall X \in S\ (1 \in X \Leftrightarrow 3 \in X).$$

(Proof shown below.)

1. For each $X \in S$, let $P(X) \equiv 1 \in X \Leftrightarrow 3 \in X$.
2. (Basis step)
   2.1. As $1,3 \in A$ and $1,3 \in B$, we know $1 \in A \Leftrightarrow 3 \in A$ and $1 \in B \Leftrightarrow 3 \in B$.
   2.2. So $P(A)$ an $P(B)$ are true.
3. (Inductive step)
   3.1. Let $X, Y \in S$ such that $P(X)$ and $P(Y)$ are true, i.e.,
   $1 \in X \Leftrightarrow 3 \in X$ and $1 \in Y \Leftrightarrow 3 \in Y$.
   3.2. Case 1: If $1,3 \in X$ and $1,3 \in Y$, then
      3.2.1. $1,3 \in X \cap Y$ and $1,3 \in X \cup Y$ and $1,3 \notin X \setminus Y$;
      3.2.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
      3.2.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
   3.3. Case 2: If $1,3 \in X$ and $1,3 \notin Y$, then
      3.3.1. $1,3 \notin X \cap Y$ and $1,3 \in X \cup Y$ and $1,3 \in X \setminus Y$;
      3.3.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
      3.3.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
   3.4. Case 3: If $1,3 \notin X$ and $1,3 \in Y$, then
      3.4.1. $1,3 \notin X \cap Y$ and $1,3 \in X \cup Y$ and $1,3 \notin X \setminus Y$;
      3.4.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
      3.4.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
   3.5. Case 4: If $1,3 \notin X$ and $1,3 \notin Y$, then
      3.5.1. $1,3 \notin X \cap Y$ and $1,3 \notin X \cup Y$ and $1,3 \notin X \setminus Y$;
      3.5.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
      3.5.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
   3.6. Hence $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true in all cases.
4. It follows that $\forall X \in S\ P(X)$ is true by structural induction over S.