CS1231S: Discrete Structures

Tutorial #8: Cardinality and Revision

(Week 10: 21 – 25 October 2024)

I. Discussion Questions

D1 Is the set of perfect squares $\{0,1,4,9,16,\ldots\}$ countable? Prove or disprove it.

D2. Aiken spoke about a set being "uncountable and infinite". Dueet commented that Aiken must have meant "uncountably infinite." Comment on what Aiken and Dueet said.

D3. [AY2021/22 Semester 2 Exam Multiple-Response Question].
Which of the following sets are countable?

A. The set A of all points in the plane with rational coordinates.
B. The set B of all infinite sequences of integers.
C. The set C of all functions $f: \{0,1\} \to \mathbb{N}$.
D. The set D of all functions $f: \mathbb{N} \to \{0,1\}$.
E. The set E of all 2-element subsets of $\mathbb{N}$.

II. Tutorial Questions

1. In lecture example #3, we showed that $\mathbb{Z}$ is countable by defining a bijection $f: \mathbb{Z}^+ \to \mathbb{Z}$ as follows:

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even;} \\ -\dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The above is based on the definition $\aleph_0 = |\mathbb{Z}^+|$. Suppose we adopt the definition $\aleph_0 = |\mathbb{N}|$ instead, define a bijection $g: \mathbb{N} \to \mathbb{Z}$ using a <u>single-line formula</u> to show that $\mathbb{Z}$ is countable.

2. Let B be a countably infinite set and C a finite set. Show that $B \cup C$ is countable

(a) by using the sequence argument;

(b) by defining a bijection $g: \mathbb{N} \to B \cup C$.

3. Recall the definition of $\bigcup_{i=m}^{n} A_i$ in Tutorial 3.

(a) Consider this claim:

"Suppose $A_1, A_2, \cdots$ are finite sets. Then $\bigcup_{i=1}^{n} A_i$ is finite for any $n \geq 2$."

The above statement is true. However, consider the following "proof":

"We will prove by induction on n. Since A_1 and A_2 are finite, then $A_1 \cup A_2$ is finite, so the claim is true for $n = 2$. Now suppose the claim is true for $n = k$, so $\bigcup_{i=1}^{k} A_i$ is finite. Let $A_{k+1} = \emptyset$. Then $\bigcup_{i=1}^{k+1} A_i = (\bigcup_{i=1}^{k} A_i) \cup A_{k+1} = \bigcup_{i=1}^{k} A_i$ which is finite by the induction hypothesis, so the claim is true for $n = k + 1$. Therefore, the claim is true for all $n \geq 2$."

What is wrong with this "proof"?

(b) Disprove the following: "Suppose $A_1, A_2, \cdots$ are finite sets. Then $\bigcup_{k=1}^{\infty} A_k$ is finite."
[The point here is: induction takes you to any finite n, but <u>not</u> to infinity.]

4. Suppose $A_1, A_2, A_3, \cdots$ are countable sets.
   (a) Prove, by induction, that $\bigcup_{i=1}^{n} A_i$ is countable for any $n \in \mathbb{Z}^+$.
   (b) Does (a) prove that $\bigcup_{i=1}^{\infty} A_i$ is countable?

5. Let S_i be a countably infinite set for each $i \in \mathbb{Z}^+$. Prove that $\bigcup_{i \in \mathbb{Z}^+} S_i$ is countable.
   [Hint: Use this theorem covered in class: $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.]

6. Let B be a (not necessarily countable) infinite set and C be a finite set.
   Define a bijection $B \cup C \to B$.

7. Let A be a countably infinite set. Prove that $\mathcal{P}(A)$ is uncountable. ($\mathcal{P}(A)$ is the power set of A.)

8. [AY2022/23 Semester 1 Exam]
   Given the following statements on any finite set A,
   (i) If R is a reflexive relation on A, then $|A| \leq |R|$.
   (ii) If R is a symmetric relation on A, then $|A| \leq |R|$.
   (iii) If R is a transitive relation on A, then $|A| \leq |R|$.

   Prove or disprove each of the statements.

9. [AY2022/23 Semester 2 Exam]
   The Fibonacci sequence F_n is defined for $n \in \mathbb{N}$ as follows:

   $$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for } n > 1.$$

   Prove by Mathematical Induction the following statement:

   $$\boldsymbol{Even(F_n) \Leftrightarrow Even(F_{n+3}), \forall n \in \mathbb{N}.}$$

   The predicate $Even(x)$ is true when the integer x is even and false otherwise. You may use the following fact in your proof:

   Fact 1: $Even(x+y) \Leftrightarrow \big(Even(x) \Leftrightarrow Even(y)\big)$

10. [AY2022/23 Semester 2 Exam]
    (a) Let $\sim$ be an equivalence relation on X and let $g : X \to Y$ be a function such that

    $$g(a) = g(b) \Leftrightarrow a \sim b \quad \forall a, b \in X.$$

    Prove or disprove the following statement:

    The following function f is well-defined:

    $$f : X/\!\sim\, \to Y \text{ given by the formula } f([x]) = g(x)\ \forall x \in X.$$

    (b) If the function f in part (a) above is well-defined, prove or disprove whether function f is injective or not injective.

    If the function f in part (a) above is not well-defined, would changing the function g in part (a) to

    $$g(a) = g(b) \Leftrightarrow a \nsim b \quad \forall a, b \in X. \text{ (Note: } \nsim \text{ is the negation of } \sim\text{.)}$$

    make the function f in part (a) well-defined? Prove or disprove.

    (c) Given the following bijections f and g on the set $A = \{1,2,3,4\}$,

    $$f = \{(1,2), (2,4), (3,1), (4,3)\};$$
    $$g = \{(1,4), (2,1), (3,2), (4,3)\}.$$

    Find the order of f, g and $(f^{-1} \circ g)$.