CS1231S: Discrete Structures

Counting and Probability II and Graphs I

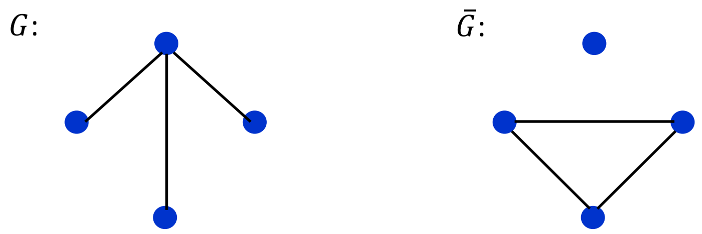
(Week 12: 4 – 8 November 2024)

Answers

II. Definitions

Definition 1. If G is a simple graph, the ***complement*** of G, denoted $\bar{G}$, is obtained as follows: the vertex set of $\bar{G}$ is identical to the vertex set of G. However, two distinct vertices v and w of $\bar{G}$ are connected by an edge if and only if v and w are not connected by an edge in G.

The figure below shows a graph G and its complement $\bar{G}$.



A graph G and its complement $\bar{G}$.

Definition 2. A ***self-complementary*** graph is isomorphic with its complement.

Definition 3. A simple circuit (cycle) of length three is called a ***triangle***.

III. Tutorial Questions

1. Find the term independent of x in the expansion of $\left(2x^2 + \frac{1}{x}\right)^9$.

Answer:

Recall the Binomial Theorem:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b^1 + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

We have $a = 2x^2, b = \frac{1}{x}, n = 9$.

The general term in the expansion of $(a+b)^n$ is given by:

$$\binom{n}{r} a^{n-r}b^r = \binom{9}{r} (2x^2)^{9-r} \left(\frac{1}{x}\right)^r = \binom{9}{r} 2^{9-r} \cdot x^{18-2r} \cdot x^{-r} = \binom{9}{r} 2^{9-r} \cdot x^{18-3r}$$

For this term to be independent of x, we must have $18 - 3r = 0$, or $r = 6$.

Therefore, the term independent of x is

$$\binom{9}{6} 2^{9-6} = 84 \times 2^3 = \mathbf{672}$$

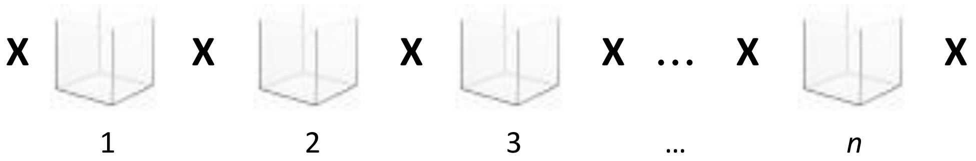
2. Let's revisit Tutorial #9 Question 4:

   Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   Last week, the answer given was

   For k $(1 \le k \le n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways. Therefore, total number of ways is $\sum_{k=1}^{n}(n-k+1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?



   Answer:

   The task is similar to choosing two out of the $n+1$ crosses to mark the start and end of the consecutively numbered boxes that contain white balls.

   This is $\binom{n+1}{2}$, which is also equal to $n(n+1)/2$.

3. [AY2020/21 Semester 2 Exam Question]
   On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.

   (i) What is the probability of rolling a 6? Write your answer as a single fraction.

   (ii) If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction.

   Answers:

Probability of $\mathbf{\frac{1}{9}}$ to roll a 1, 2 or 3; probability of $\mathbf{\frac{2}{9}}$ to roll a 4, 5 or 6.

Maximum of 1: 1 way → (1,1) with probability $\frac{1}{81}$.

Maximum of 2: 3 ways → (1,2), (2,1), (2,2) with probability $\frac{3}{81}$.

Maximum of 3: 5 ways → (1,3)x2, (2,3)x2, (3,3) with probability $\frac{5}{81}$.

Maximum of 4: 7 ways → (1,4)x2, (2,4)x2, (3,4)x2, (4,4) with probability $\frac{12}{81} + \frac{4}{81} = \frac{16}{81}$.

Maximum of 5: 9 ways → (1,5)x2, (2,5)x2, (3,5)x2, (4,5)x2, (5,5) with probability $\frac{12}{81} + \frac{12}{81} = \frac{24}{81}$.

Maximum of 6: 11 ways → (1,6)x2, (2,6)x2, (3,6)x2, (4,6)x2, (5,6)x2, (6,6) with probability $\frac{12}{81} + \frac{20}{81} = \frac{32}{81}$.

Expected value = $\left(\frac{1}{81} \times 1\right) + \left(\frac{3}{81} \times 2\right) + \left(\frac{5}{81} \times 3\right) + \left(\frac{16}{81} \times 4\right) + \left(\frac{24}{81} \times 5\right) + \left(\frac{32}{81} \times 6\right) = \mathbf{\frac{398}{81}}$.

4. An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls?

Answer:

Let $2_a, 2_b$ denote the two balls with the number 2, and $8_a, 8_b$ the two balls with the number 8.

The list below shows the sums of the numbers on the balls in each set:

- Sum of 3: $\{1, 2_a\}, \{1, 2_b\}$.
- Sum of 4: $\{2_a, 2_b\}$.
- Sum of 9: $\{1, 8_a\}, \{1, 8_b\}$.
- Sum of 10: $\{2_a, 8_a\}, \{2_a, 8_b\}, \{2_b, 8_a\}, \{2_b, 8_b\}$.
- Sum of 16: $\{8_a, 8_b\}$.

Expected value = $3 \times \left(\frac{2}{10}\right) + 4 \times \left(\frac{1}{10}\right) + 9 \times \left(\frac{2}{10}\right) + 10 \times \left(\frac{4}{10}\right) + 16 \times \left(\frac{1}{10}\right) = \mathbf{8.4}$.

Or, we could compute the expected value of the number when one ball is picked:
$1 \times \left(\frac{1}{5}\right) + 2 \times \left(\frac{2}{5}\right) + 8 \times \left(\frac{2}{5}\right) = 4.2$.
Then apply linearity of expected value to get $2 \times 4.2 = \mathbf{8.4}$.

5. [AY2021/22 Semester 2 Exam Question]
A rare disease broke out in a city with a prevalence of 0.1%, that is, it affects 1 out of every 1000 persons. A quick test kit has been developed that has a sensitivity of 85%, which is the probability that a person with the rare disease is tested positive. Among those who took the test, 10% of the time it came out positive. Write your answers correct to 3 significant figures.

(a) Divoc has shown symptoms of the disease. Should he be tested positive, what is the probability that he actually has the disease?

(b) What is the probability of a false positive result, that is, a person does not have the disease but is tested positive?

Answers:

$P(Disease) = 0.001;\ P(+|Disease) = 0.85;\ P(+) = 0.1.$

(a) $P(Disease \mid +) = \frac{P(+|\,Disease) \cdot P(Disease)}{P(+)} = \frac{0.85 \times 0.001}{0.1} = \mathbf{0.00850}$ or $\mathbf{0.850\%}$.

(b) $P(+ \mid \overline{Disease}) = \frac{P(\overline{Disease} \mid +) \cdot P(+)}{P(\overline{Disease})} = \frac{(1 - 0.00850) \times 0.1}{0.999} = \mathbf{0.0992}$ or $\mathbf{9.92\%}$.

6. [AY2015/16 Semester 1 Exam Question]
Let $A = \{1, 2, 3, 4\}$. Since each element of $P(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. ($P(S)$ denotes the powerset of S.)

Assuming each relation in $P(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

Can you generalize your answer to any set A with n elements?

Answers:
(a) 1/16 (b) 1/64

For the general case, let $|A| = n$. In general, we need to consider all possible n^2 pairs. Thus, the total number of possible combinations, i.e. the number of elements in $P(A \times A)$, is 2^{n^2}.

(a) To count the number of reflexive relations, note that all the pairs (a, a) for all $a \in A$ must be in the relation, so these n pairs are fixed. We are then free to choose to include or not any of the $(n^2 - n)$ remaining pairs. This gives us:

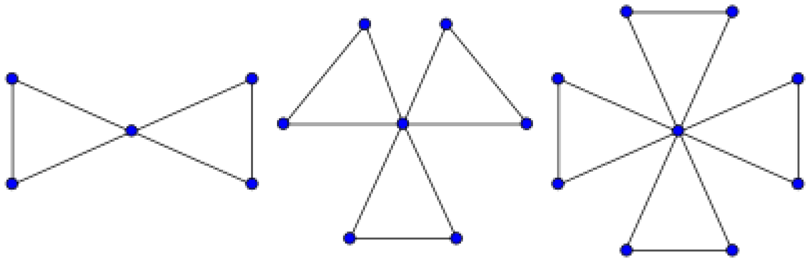
$$\frac{2^{n^2-n}}{2^{n^2}} = \frac{1}{2^n}$$

(b) For symmetric relations, if some pair (a, b) is in the relation, then (b, a) must also be in the relation. Hence, either (a, b) and (b, a) are both included in the relation, or both are not. This gives us $\frac{n^2+n}{2}$ pairs to choose to include or not, giving us:

$$\frac{2^{\frac{n^2+n}{2}}}{2^{n^2}} = \frac{1}{2^{\frac{n^2-n}{2}}}$$

(Refer to the slides uploaded for more details.)

7. The **friendship graph** (or **Dutch windmill graph** or ***n*-fan**) F_n is a planar, undirected graph with $2n + 1$ vertices and $3n$ edges. (Ref: https://en.wikipedia.org/wiki/Friendship_graph)



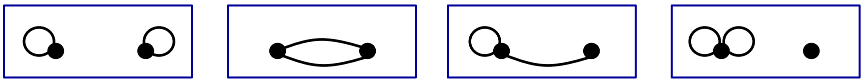
F_2 F_3 F_4

Is the friendship graph Eulerian? Is it Hamiltonian?

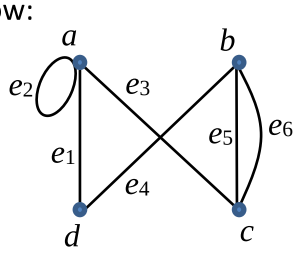
Answer:
The friendship graph is Eulerian as all vertices have even degree.
Except for F_1, it is not Hamiltonian. For F_n where $n \geq 2$, to get from a vertex in a triangle to a vertex in another triangle, we need to pass through the centre vertex, hence visiting the centre vertex more than once.

8. [Past year's exam question]
How many non-isomorphic undirected graphs with two vertices and two edges are there? Draw them.

Answer:
4 (Question doesn't say that the graphs are simple.)



9. Given the graph shown below:



(a) Give the adjacency matrix A for the graph, with vertices in the order a, b, c, d.

(b) Compute A^0, A^2 and A^3.

(c) How many walks of length 2 are there from a to b? From c to itself? List out all the walks.

(d) How many walks of length 3 are there from a to c? List out all the walks.

Answers:

(a)
$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(b) $A^0 = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; $A^2 = \begin{bmatrix} 3 & \mathbf{3} & 1 & 1 \\ 3 & 5 & 0 & 0 \\ 1 & 0 & \mathbf{5} & 3 \\ 1 & 0 & 3 & 2 \end{bmatrix}$; $A^3 = \begin{bmatrix} 5 & 3 & \mathbf{9} & 6 \\ 3 & 0 & 13 & 8 \\ 9 & 13 & 1 & 1 \\ 6 & 8 & 1 & 1 \end{bmatrix}$

(c) There are **3** walks of length 2 from a to b:
$\langle a \rightarrow c \overset{e_5}{\rightarrow} b \rangle, \langle a \rightarrow c \overset{e_6}{\rightarrow} b \rangle, \langle a \rightarrow d \rightarrow b \rangle.$

There are **5** walks of length 2 from c to itself:
$\langle c \rightarrow a \rightarrow c \rangle, \langle c \overset{e_5}{\rightarrow} b \overset{e_5}{\rightarrow} c \rangle, \langle c \overset{e_5}{\rightarrow} b \overset{e_6}{\rightarrow} c \rangle, \langle c \overset{e_6}{\rightarrow} b \overset{e_5}{\rightarrow} c \rangle, \langle c \overset{e_6}{\rightarrow} b \overset{e_6}{\rightarrow} c \rangle.$

(d) There are **9** walks of length 3 from a to c:

1 way via a first: $\langle a \rightarrow a \rightarrow a \rightarrow c \rangle.$
0 way via b first.
5 ways via c first: $\langle a \rightarrow c \rightarrow a \rightarrow c \rangle, \langle a \rightarrow c \overset{e_5}{\rightarrow} b \overset{e_5}{\rightarrow} c \rangle, \langle a \rightarrow c \overset{e_5}{\rightarrow} b \overset{e_6}{\rightarrow} c \rangle,$
$\langle a \rightarrow c \overset{e_6}{\rightarrow} b \overset{e_5}{\rightarrow} c \rangle, \langle a \rightarrow c \overset{e_6}{\rightarrow} b \overset{e_6}{\rightarrow} c \rangle.$
3 ways via d first: $\langle a \rightarrow d \rightarrow a \rightarrow c \rangle, \langle a \rightarrow d \rightarrow b \overset{e_5}{\rightarrow} c \rangle, \langle a \rightarrow d \rightarrow b \overset{e_6}{\rightarrow} c \rangle.$

10. A lady hosted a party of n ($n \geq 2$) people (including herself). At the party, various friends met and some of them shook hands with each other. The thoughtful host made sure that she shook hands with everyone at the party.

    Prove that there are at least two people who have shaken hands the same number of times.

    Answer:

    Model this with a graph $G = (V, E)$ where each person is a vertex in V and there is an undirected edge $\{x, y\} \in E$ whenever x and y shook hand at the party.

    1. Let h be the host vertex. Then $deg(h) = n - 1$. (given by the problem)
    2. For other vertices v, $deg(v) > 0$. (shook hand with host *h*)
    3. So all n vertices have $deg(v)$ in the range $[1..(n-1)]$.
    4. Hence, at least two vertices have the same degree. (by the Pigeonhole Principle)

    Hence at least two people have the *same* number of handshakes.

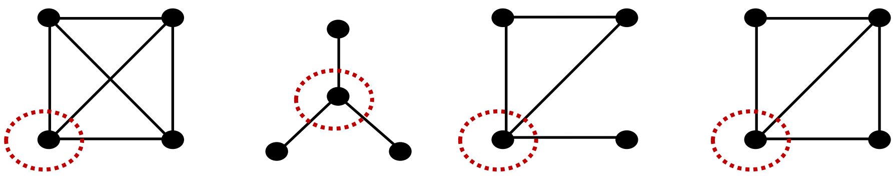
11. [AY2021/22 Semester 1 Exam]
    A set of vertices, D, in an undirected simple graph is said to be a ***dominating set*** if every vertex not in D is adjacent to at least one vertex in D. A ***minimal dominating set*** is a dominating set such that none of its proper subsets are dominating sets.

    (a) Draw two non-isomorphic simple graphs with four vertices that have minimal dominating sets of size one. Highlight the vertices in the minimal dominating set in your graphs.

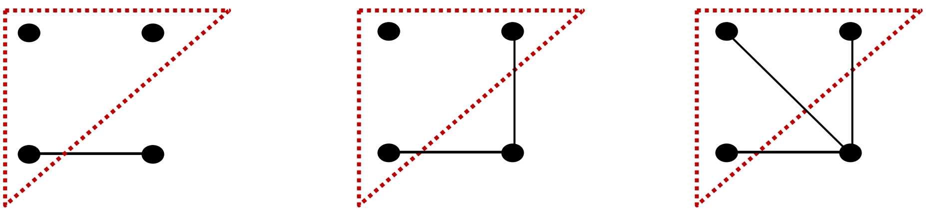
    (b) Draw two non-isomorphic simple graphs with four vertices that have minimal dominating sets of size three. Highlight the vertices in the minimal dominating set in your graphs.

    Answers:

    (a) We accept any two of the following answers. Dominating set shown in dotted circle.



    (b) We accept any two of the following answers. Dominating set shown in dotted triangle.



12. Let G be any simple graph with 6 vertices. Prove that G or its complementary graph $\bar{G}$ contains a triangle. (This is similar to the problem of showing that in any group of 6 people, there must be either 3 mutual friends or 3 mutual strangers. But for this question, please use the graph formulation.)

Answer:

1. Let G be a simple graph with 6 vertices. (eg: Fig. 10a)
2. In K_6, the complete graph on 6 vertices, we colour the edges in G **black**, and those not in G red. Namely, the red edges are those in the complement graph $\bar{G}$. (See Figs 10a-c.)

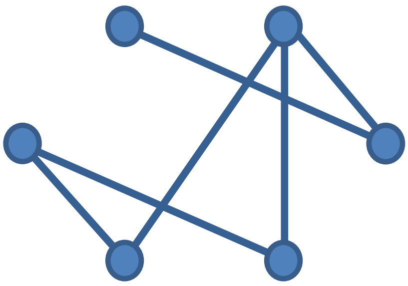


Fig. 10a. The graph

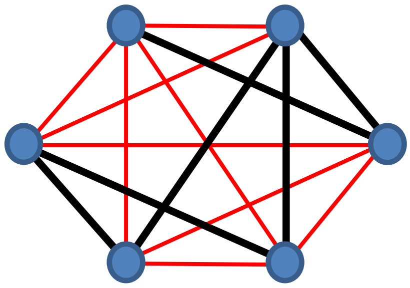


Fig. 10b. The coloured K_6 graph.

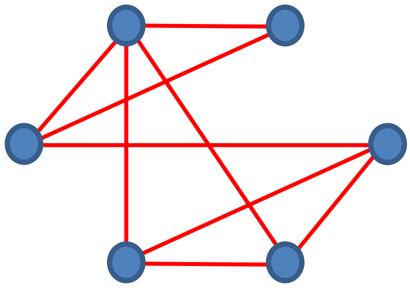


Fig. 10c. The complementary graph

3. In this coloured K_6, the edges in G are coloured black while those in $\bar{G}$ are red. We want to prove that this coloured K_6 contains a black triangle (cycle of length 3) or a red triangle.
4. Let v be an arbitrary vertex of the coloured K_6.

   4.1. There are 5 edges incident to v (since K_6 is a complete graph), coloured black or red.

   4.2. Therefore, (at least) 3 them have the same colour C, by the Generalized PHP. (In Figure 10d, 3 edges incident to v are red.)

   4.3. WLOG, let the colour C be red and the red edges be $\{v, u_1\}$, $\{v, u_2\}$, $\{v, u_3\}$. (Fig 10d).

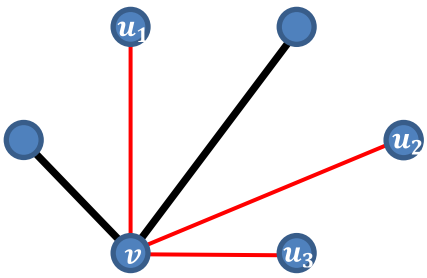


Fig. 10d

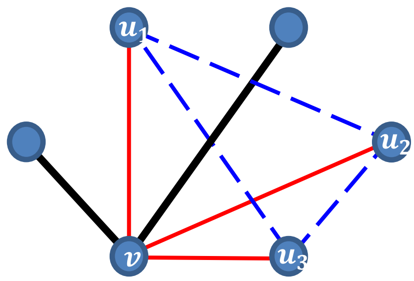


Fig. 10e

   4.4. **Case 1:** If at least one of the edges $\{u_1, u_2\}$, $\{u_2, u_3\}$ or $\{u_1, u_3\}$ is red.
   If $\{u_x, u_y\}$ is the edge that is red, then we have a red triangle $\{v, u_x, u_y\}$. (See Fig. 10e.) This is a triangle in $\bar{G}$.

   4.5. **Case 2:** If all the three edges $\{u_1, u_2\}$, $\{u_2, u_3\}$ or $\{u_1, u_3\}$ are black.
   Then, we have a black triangle $\{u_1, u_2, u_3\}$. (See Fig. 10e.) This is a triangle in G.

5. Therefore, G or $\bar{G}$ contains a triangle.

   [Note: For practice, students may want to also write out the case for "WLOG, let C be black". And also modify Figures 10d and 10e, accordingly.]