CS1231S: Discrete Structures

Tutorial #11: Graph II and Tree

(Week 13: 11 – 15 November 2024)

Answers

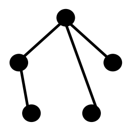
II. Tutorial Questions

You can freely use the following lemma (without proof). The proof is left as an optional exercise.

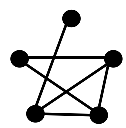
Lemma 10.5.5. Let G be a simple, undirected graph. If there are two distinct paths from a vertex v to a different vertex w, then G contains a cycle (and hence G is cyclic).

1. (a) For the following graph G, draw its complement graph $\bar{G}$.

G:



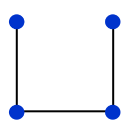
Answer: $\bar{G}$:



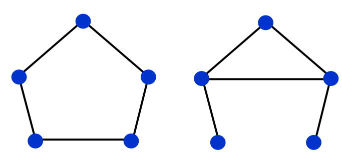
(b) Consider simple undirected graphs on n vertices. Draw all self-complementary graphs with n vertices (for $n = 3, 4, 5, 6$), or justify why there are none.

Answers

For $n = 4$:



For $n = 5$:

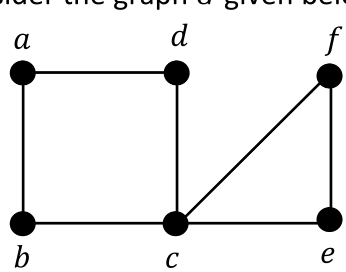


For $n = 3$, K_3 has 3 edges. Cannot be evenly divided into 2 equal halves.

For $n = 6$, K_6 has 15 edges. Cannot be evenly divided into 2 equal halves.

2. Consider the graph G given below. How many spanning trees of G are there?

G:



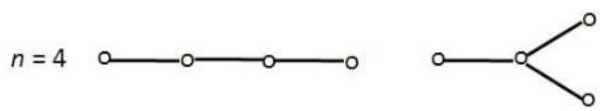
There are 2 cycles $C_1 = \{a, b, c, d\}$ and $C_2 = \{c, e, f\}$.
They are edge-disjoint (no common edge).
We need to remove 1 edge from each cycle:
4 choice for C_1 and 3 choice for C_2.
Product rule: Total ways = $4 \times 3 = \mathbf{12}$.

3. (a) Draw all non-isomorphic trees with n vertices, $n = 1,2,3,4$.

(b) For each non-isomorphic tree in part (a), how many isomorphic copies are there?

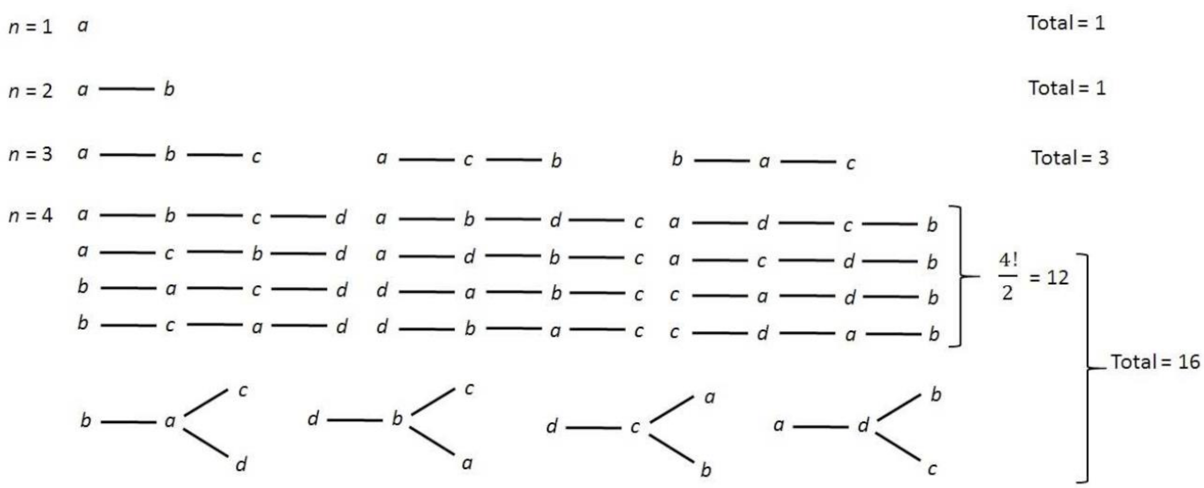
Answers:

(a)



	Total	# of isomorphic graphs
$n=1$	Total = 1	1
$n=2$	Total = 1	1
$n=3$	Total = 1	3
$n=4$	Total = 2	12, 4

(b) For each non-isomorphic tree above, we label the vertices and determine how many different ways to permute the labels.



For $n = 3$, there are $3!/2 = 3$ different ways to permute the labels of the graph.
For $n = 4$, there are $4!/2 = 12$ different ways to permute the labels of the a-b-c-d path, and 4 ways to select the middle vertex for the other non-isomorphic graph.

4. (a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is connected, then $|E| \geq |V| - 1$.

(b) Is the converse true?

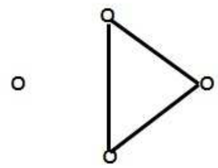
Proposition 10.7.1
1. Every connected graph has a spanning tree.
2. Any two spanning trees for a graph have the same number of edges.

Answers:

(a)
1. Suppose that $G = (V, E)$ is connected.
2. Then G has a spanning tree $T = (V, F)$, where $F \subseteq E$. (by Proposition 10.7.1)
3. Then $|F| = |V| - 1$. (by Theorem 10.5.2)
4. Thus, $|E| \geq |F| = |V| - 1$.

Theorem 10.5.2
Any tree with n vertices ($n > 0$) has $n - 1$ edges.

(b)



$G' = (V', E')$

Converse is NOT true.
This graph $G' = (V', E')$ has $(|V'| - 1)$ edges, but the graph is not connected.

5. (a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is acyclic, then $|E| \leq |V| - 1$.

 (b) Is the converse true?

Answers:

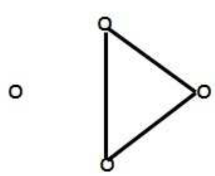
(a)

1. Suppose that $G = (V, E)$ is acyclic.
2. Let the connected components in $G = (V, E)$ be
 $H_1 = (V_1, E_1),\ H_2 = (V_2, E_2), \ldots,\ H_k = (V_k, E_k)$, where $k \geq 1$.
 - 2.1. Each $H_i = (V_i, E_i)$, is connected. (definition of connected components)
 - 2.2. Each $H_i = (V_i, E_i)$, is also *acyclic*. (since G is acylic)
 - 2.3. Hence, each $H_i = (V_i, E_i)$ is a tree. (definition of tree)
 - 2.4. Therefore, $|E_i| = |V_i| - 1$, for $i = 1,2, \ldots, k$. (by Theorem 10.5.2)
3. So, $|E| = |E_1| + |E_2| + \cdots |E_k|$ (by Addition Rule)
 $= (|V_1| - 1) + (|V_2| - 1) + \cdots + (|V_k| - 1) = |V| - k$. (by 2.4)
4. Hence, $|E| = |V| - k$, and therefore $|E| \leq |V| - 1$ since $k \geq 1$.

[**Note to students**: Given that G is any simple, undirected graph that is acyclic, we do not have any good leverage/property to use in our proof.

But, we have many theorems dealing with connected graphs.
So, one way is to consider Case 1: G is connected and Case 2: G is not connected.
When G is not connected, we can consider the k *connected* components of G.
In this problem, Case 1 just happen to be a special case where $k = 1$. Hence, Case 1 is just "absorbed" into Case 2.]

(b)



$G' = (V', E')$

Converse is NOT true.
This graph $G' = (V', E')$ has $(|V'| - 1)$ edges, but cyclic (and not connected).

6. Let $G = (V, E)$ be a simple, undirected graph. Prove that G is a tree if and only if there is exactly one path between every pair of vertices.

> **Lemma 10.5.5**
> Let G be a simple, undirected graph. If there are two distinct paths from a vertex v to a different vertex w, then G contains a cycle (and hence G is cyclic).

Answer:

(⇒)

1. Let G be a tree.
2. As G is connected, there is a path between any pair of vertices.
3. If some pair of vertices has two or more paths connecting them, then G is cyclic (by Lemma 10.5.5).
4. This contradicts G is a tree as trees are acyclic.
5. Therefore, every pair of vertices has exactly one path between them.

(⇐)

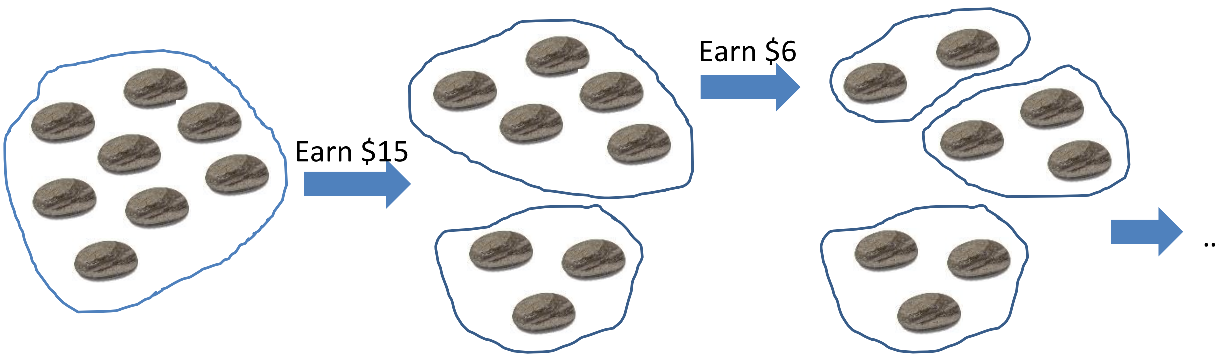
1. Let G be a simple, undirected graph.
2. Suppose there is exactly one path between every pair of vertices, then G is connected (by definition of connectedness).
3. Suppose G is cyclic, then there is a cycle C in G.
   3.1. Let x and y be two distinct vertices in the cycle C.
   3.2. Then there are two paths connecting x and y in the cycle C, which contradicts line 2.
   3.3. Hence, G must be acyclic.
4. Therefore, G is tree (by lines 2 and 3.3, and definition of a tree).

7. (AY2017/18 Semester 1 Exam Question)
Suppose you are given a pile of stones. At each step, you can separate a pile of k stones into two piles of k_1 and k_2 stones. (Obviously, $k_1 + k_2 = k$.) On doing this, you earn \$$(k_1 \times k_2)$.

What is the maximum amount of money you can earn at the end if you start with a pile of n stones? Explain your answer.

The diagram below illustrates the (incomplete) process of separating a pile of 8 stones.

(This problem may be solved without using graph theory, but here we want to model it as a graph problem.)



Answer:

One can earn at most \$$n(n-1)/2$. (why?)

Represent each stone as a vertex in a graph. Two vertices are connected with an edge if the stones they represent are in the same pile. In the beginning, we have a graph of n vertices with $n(n-1)/2$ edges as it is a K_n complete graph.

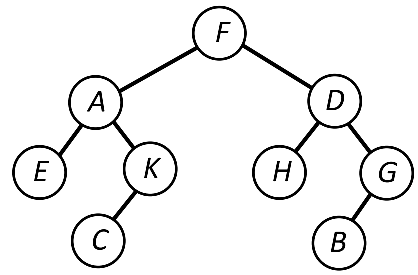
Separating a pile of k stones into two piles of k_1 and k_2 stones corresponds to removing some edges from the graph. The number of edges removed is exactly $k_1 \times k_2$.

The maximum amount of money one can earn is when the stones are separated into piles with single stone, that is, when all the edges are removed. Since the number of edges in a K_n complete graph is $n(n-1)/2$, this is the maximum amount one can earn.

8. (a) A binary tree T_1 has 9 vertices. The in-order and pre-order traversals of T_1 are given below. Draw the tree T_1 and give its post-order traversal.

In-order: E A C K F H D B G
Pre-order: F A E K C D H G B

Answer:



Post-order: E C K A H B G D F

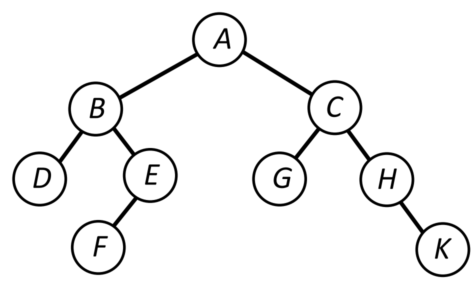
Strategy:

- The first vertex in pre-order traversal is root of tree. (In our example, this is vertex F.)
- Find vertex F in in-order traversal. All vertices appearing before F (in-order) belong to left subtree; All vertices appearing after F (in-order) belong to right subtree;
- Recursively apply procedure to left subtree and right subtree.

(b) A binary tree T_2 has 9 vertices. The in-order and post-order traversals of T_2 are given below. Draw the tree T_2 and give its pre-order traversal.

In-order: D B F E A G C H K
Post-order: D F E B G K H C A

Answer:



Pre-order: A B D E F C G H K.

Strategy:

Now last vertex in post-order traversal is root of tree. (In our example, this is vertex A.)

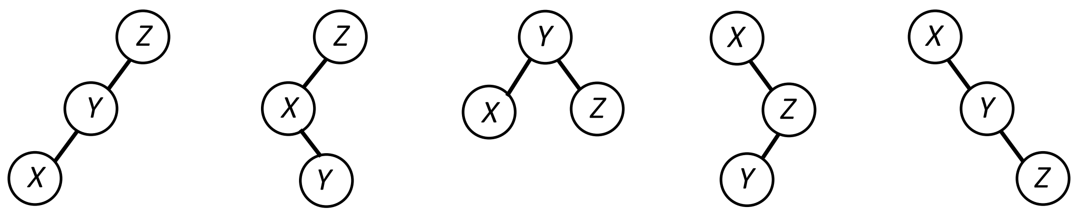
Apply a similar method.

9. (a) Draw all possible binary trees with 3 vertices X, Y and Z with in-order traversal: $X\ Y\ Z$.

(b) Draw all possible binary trees with 4 vertices A, B, C and D with in-order traversal: $A\ B\ C\ D$.

Answers:

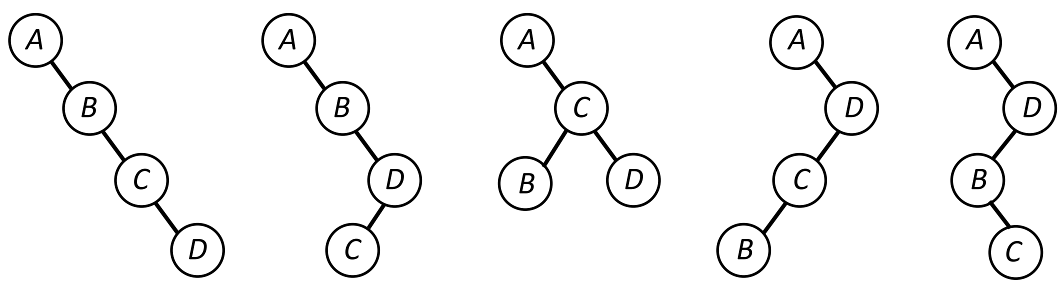
(a) 5 possible binary trees (3 vertices: X, Y, Z)



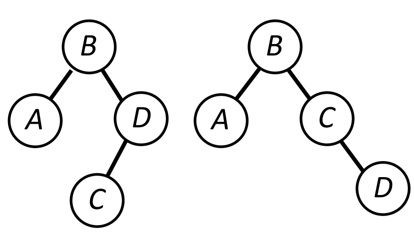
(b) 14 possible binary trees (4 vertices A, B, C, D)

(Strategy: Fix the root of the binary tree; then we know #verticess in left and right subtrees.)

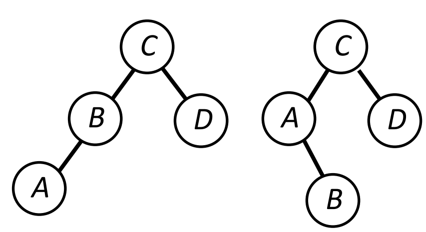
Rooted at *A*: 5



Rooted at *B*: 2



Rooted at *C*: 2



Rooted at *D*: 5 (not shown)

[Optional, for the FUN of it] The above strategy also gives hint to a recurrence relation, that when solved gives the general solution for larger n. It is an example of a *convolution recurrence*.

$$C_{n+1} = C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \cdots C_kC_{n-k} + C_nC_0$$

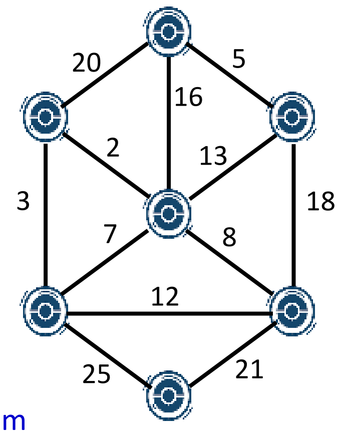
This sequence is called the Catalan's number sequence: 1, 2, 5, 14, 42, 132, …

The general form for the Catalan's number is $C_n = \frac{1}{(n+1)}\binom{2n}{n}$.]

10. (*Modified from* AY2016/17 Semester 1 Exam Question)

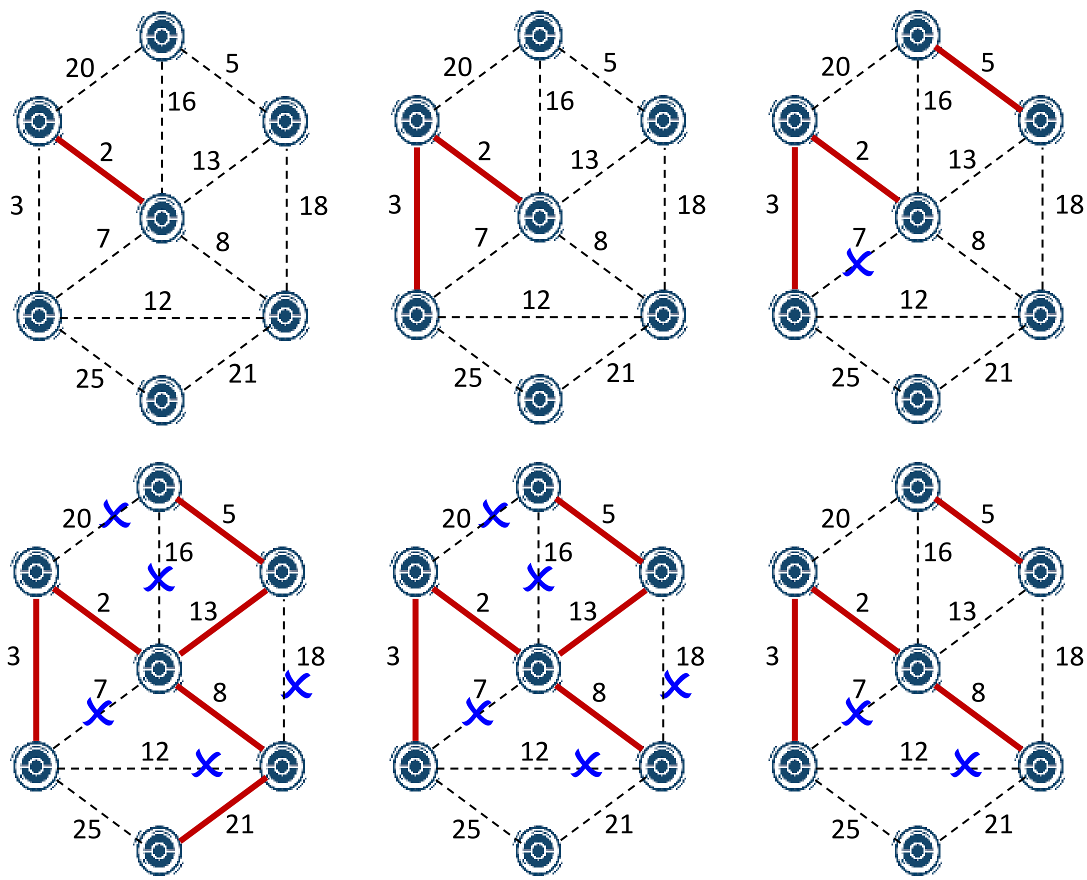
The figure below shows a graph where the vertices are Pokestops. Using either Kruskal's algorithm or Prim's algorithm, find its minimum spanning tree (MST). If you use Prim's algorithm, you must start with the top-most vertex.

Indicate the order of the edges inserted into the MST in your answer.



Answer: Kruskal's algorithm

Edges:
2
3
5
7 ×
8
12 ×
13
16 ×
18 ×
20 ×
21
25



Prim’s algorithm.