CS1231S: Discrete Structures

Tutorial #11: Graph II and Tree

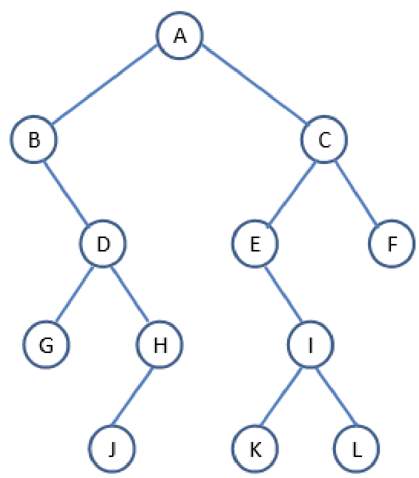
(Week 13: 11 – 15 November 2024)

I. Discussion Questions

D1. For any simple connected graph with n $(n > 0)$ vertices, what is the minimum and maximum number of edges the graph may have?

D2. (AY2016/17 Semester 1 Exam Question)
How many simple graphs on 3 vertices are there?
In general, how many simple graphs on n $(n > 1)$ vertices are there?

D3. Given the following binary tree, write the pre-order, in-order, and post-order traversals of its vertices.



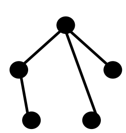
II. Tutorial Questions

You can freely use the following lemma (without proof). The proof is left as an optional exercise.

Lemma 10.5.5. Let G be a simple, undirected graph. Then if there are two distinct paths from a vertex v to a different vertex w, then G contains a cycle (and hence G is cyclic).

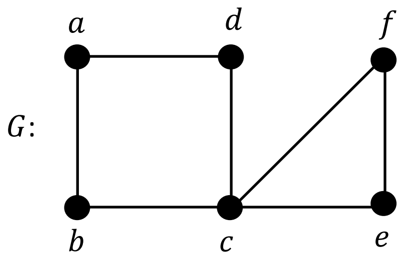
1. (a) For the following graph G, draw its complement graph $\bar{G}$.

G:



(b) Consider simple graphs on n vertices. Draw all self-complementary graphs with n vertices (for $n = 3, 4, 5, 6$), or justify why there are none.

2. Consider the graph G given below. How many spanning trees of G are there?



3. (a) Draw all non-isomorphic trees with n vertices, $n = 1,2,3,4$.

   (b) For each non-isomorphic tree in part (a), how many isomorphic copies are there?

4. (a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is connected, then $|E| \geq |V| - 1$.

   (b) Is the converse true?

5. (a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is acyclic, then $|E| \leq |V| - 1$.

   (b) Is the converse true?

6. Let $G = (V, E)$ be a simple, undirected graph. Prove that G is a tree if and only if there is exactly one path between every pair of vertices.

7. (AY2017/18 Semester 1 Exam Question)
   Suppose you are given a pile of stones. At each step, you can separate a pile of k stones into two piles of k_1 and k_2 stones. (Obviously, $k_1 + k_2 = k$.) On doing this, you earn $\$(k_1 \times k_2)$.

   What is the maximum amount of money you can earn at the end if you start with a pile of n stones? Explain your answer.

   The diagram below illustrates the (incomplete) process of separating a pile of 8 stones.

   (This problem may be solved without using graph theory, but here we want to model it as a graph problem.)

8. (a) A binary tree T_1 has 9 vertices. The in-order and pre-order traversals of T_1 are given below. Draw the tree T_1 and give its post-order traversal.

In-order: E A C K F H D B G
Pre-order: F A E K C D H G B

(b) A binary tree T_2 has 9 vertices. The in-order and post-order traversals of T_2 are given below. Draw the tree T_2 and give its pre-order traversal.

In-order: D B F E A G C H K
Post-order: D F E B G K H C A

9. (a) Draw all possible binary trees with 3 vertices X, Y and Z with in-order traversal: $X\ Y\ Z$.

(b) Draw all possible binary trees with 4 vertices A, B, C and D with in-order traversal: $A\ B\ C\ D$.

10. (*Modified from* AY2016/17 Semester 1 Exam Question)
The figure below shows a graph where the vertices are Pokestops. Using either Kruskal's algorithm or Prim's algorithm, find its minimum spanning tree (MST). If you use Prim's algorithm, you must start with the top-most vertex.

Indicate the order of the edges inserted into the MST in your answer.