MA 5219 - Logic and Foundations of Mathematics 1
Homework due in Week 10, Tuesday.

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Hand in each starred homework; 1 mark per homework (if it is correct), up to 10 marks in total for homework.

13.1* Rice's Theorem. Let $\varphi_0, \varphi_1, \ldots$ be any Gödel numbering of the partial-recursive functions with one input. Say two programs d, e are equivalent iff for all inputs x, either $\varphi_d(x)$ and $\varphi_e(x)$ are both undefined or they are both defined and equal. Now E is said to be an index-set iff for all equivalent indices d, e, either both $d, e \notin E$ or both $d, e \in E$. Rice's Theorem says that there are no recursive index-set besides $\emptyset$ and $\mathbb{N}$. For which of the following sets, one can use Rice's Theorem to show that they are not recursive (none of these sets is actually recursive):

1. $\{e : \varphi_e(0) \text{ is defined}\}$;
2. $\{e : \varphi_e(e) \text{ is defined}\}$;
3. $\{e : \varphi_e(x) \text{ is defined for infinitely many } x\}$;
4. $\{e : \text{at least one of } \varphi_e(256), \varphi_e(257) \text{ and } \varphi_e(258) \text{ is defined}\}$;
5. $\{e : \varphi_e(0) \text{ and } \varphi_e(\varphi_e(0)) \text{ are both defined}\}$.

13.2 Hierarchies of Sets. A subset $A \subseteq \mathbb{N}$ is called a Σ_n-set iff there exists an arithmetic Σ_n-formula ϕ with $A = \{n \in \mathbb{N} : (\mathbb{N}, +, \cdot, 0, 1) \models \phi(\underline{n})\}$. Similarly one defines Π_n-sets. An alternative approach is the following: An r.e. set is a Σ_1-set and a co-r.e. set is a Π_1-set. Then, a Σ_{n+1}-set is any set which can be enumerated by a function f which is recursive relative to A where A is a Σ_n-set. Here "recursive relative to A" means that f is computed by a register program which uses A as a subprogram to determine whether $A(x)$ is 1 ($x \in A$) or 0 ($x \notin A$). Use the fact that the two definitions of Σ_n-sets coincide to prove that there is no formula ϕ with one parameter such that $\phi(\underline{n})$ is true iff the n-th sentence (according to some Gödel numbering of all setences) is true in $(\mathbb{N}, +, \cdot, 0, 1)$. This gives an alternative proof of Tarski's theorem.

13.3 Nonexistence of Decidable Extensions of Q. Assume by way of contradiction that there is a theory T extending Q for which the set $\{gn(\phi) : \phi \text{ is a sentence and } \vdash_T \phi\}$ is decidable. Then show that there is is a partial-recursive $\{0,1\}$-valued function ψ such that no total recursive function f extends ψ. Produce a formula ϕ such that $\psi(n) = 0$ implies $\phi(\underline{n})$ and $\psi(n) = 1$ implies $\neg\phi(\underline{n})$. Use this to derive a contradiction with the assumptions on T. Here $gn(\phi)$ is the Gödel number of the formula ϕ.