MA 5219 - Logic and Foundations of Mathematics 1
Homework due in Week 3, Tuesday 27 August 2013.

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Hand in each homework which you want to be checked; 1 mark per each correct starred homework; up to 10 marks in total for homework - there will be more than 10 starred homeworks, so you have several chances to try.

3.1 Conjunctive and Disjunction Normalform. Make formulas in conjunctive normal form and disjunctive normal form which state that exactly 2 of the atoms p_1, p_2, p_3, p_4 are true.

3.2* **Worlds.** Recall that a world is an entity which assigns a truth-value to every atom. $W \models A$ means that the world makes the formula A true and $W \models X$ means that the world makes all formulas in the set X of formulas true. $X \models A$ means that every world which makes all formulas in X true, also makes the formula A true.
Let V and W be two different worlds, let $X = \{A :$ exactly one of $V \models A$ and $W \models A$ holds$\}$, let $Y = \{A :$ both of $V \models A$ and $W \models A$ hold$\}$. Show the following.
(a) There is an atom p_k such that either ($V \models p_k$ and $W \models \neg p_k$) or ($V \models \neg p_k$ and $W \models p_k$).
(b) If $A \in Y$ then there exist $B \in X$ and $C \in X$ with $\emptyset \models (A \leftrightarrow B \vee C)$.
(c) There are formulas $A, B \in X$ with $A \vee B \notin X$.
(d) If $A, B, C \in X$ then at least one of the formulas $A \vee B$, $A \vee C$, $B \vee C$ is in X.

3.3 Logical Implication. (a) Assume that $W \models X$ for all worlds W which make only finitely many atoms true. Show that $W \models X$ for all worlds and that X contains only tautologies.
(b) Construct a set X of formulas such that $W \models X$ is true iff W makes at most two atoms true and all others false. Note that X can be infinite.

3.4* **Proof System.** Assume that only $A + B$ is permitted to connect formulas A, B, which are built from the atoms $p_0, p_1, \ldots$ and the logical constants 0 and 1. For the ease of notation, it is assumed that formulas are written without brackets, as the operation + is associative. Is there a set of rules which is permits to prove a set of formulas Y from X whenever $X \models Y$ and X and Y are both sets of formulas of the above form and Y is finite? If so, give the set of rules; if not, explain why a set of such rules cannot exist.