MA 5219 - Logic and Foundations of Mathematics 1
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogic.html`
Homework due in Week 8, Tuesday 8 October 2013.

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Hand in each homework which you want to be checked; 1 mark per each correct starred homework; up to 10 marks in total for homework - there will be more than 10 starred homeworks, so you have several chances to try.

8.1 Logical Implication. Here let e is a constant and v, w, x, y, z are variables. Let

$$\begin{array}{ccl} X & \text{contain} & \forall x, y, z\, [x \circ (y \circ z) = (x \circ y) \circ z],\ \forall x [x \circ e = x],\ \forall x\, [x \circ x = e] \\ \alpha & \text{be} & v \circ w = w \circ v, \\ \beta & \text{be} & v \circ (v \circ v) = v \text{ and} \\ \gamma & \text{be} & v \circ (v \circ v) = e. \end{array}$$

Do $X \models \alpha$, $X \models \beta$ and $X \models \gamma$ hold? Justify for each of the formulas α, β, γ your answers.

8.2 Henkin Sets. Let the logical language $\mathcal{L}$ contain first-order formulas over variables $x_0, x_1, \ldots$ and constants $c_0, c_1, \ldots$ and one predicate symbol P. Check whether the following sets X, Y and Z are Henkin sets. Explain why you think that the corresponding sets are Henkin sets or not.

X contains for all distinct i, j the formula $c_i \neq c_j$ as well as for every k the formulas $x_k = c_k$ and $P(c_k)$.

Y contains the same formulas as X plus the formula $\forall x_0\, [P(x_0)]$.

Z contains the same formulas as X plus the formula $\exists x_0\, [\neg P(x_0)]$.

8.3* **Countable models.** Assume that X is a Henkin set and $\mathcal{A}$ is a model of X. Show that $\mathcal{A}$ has a substructure $\mathcal{B}$ which is an at most countable model of X.

8.4* **Proof System.** Use the enlarged rule system on next page obtained by adding Modus Ponens and two similar further rules for $\rightarrow$ to those of page 92 in order to prove the following tautology (where e is a constant and $\circ$ an operation symbol which could be rewritten as a function with two inputs):

$$\forall x\, [x \circ (x \circ x) = e] \rightarrow \forall x, y, z\, [x \circ (y \circ z) = (x \circ y) \circ z] \rightarrow \forall x\, [x \circ e = x] \rightarrow$$
$$\forall x\, [e \circ x = x] \rightarrow \forall y, z\, [y \circ (z \circ (y \circ (z \circ y))) = z \circ z].$$

Derivation Rules. Here are the rules from page 92 of Rautenberg's book plus the three rules for $\rightarrow$.

$$(IR) \quad \frac{}{X \vdash \alpha} \text{ where } \alpha \in X \text{ or } \alpha \text{ is of the form } t = t$$

$$(MR) \quad \frac{X \vdash \alpha}{X \cup Y \vdash \alpha}$$

$$(MP) \quad \frac{X \vdash \alpha, \alpha \rightarrow \beta}{X \vdash \beta}$$

$$(\rightarrow 1) \quad \frac{X \vdash \alpha \rightarrow \beta}{X, \alpha \vdash \beta}$$

$$(\rightarrow 2) \quad \frac{X, \alpha \vdash \beta}{X \vdash \alpha \rightarrow \beta}$$

$$(\wedge 1) \quad \frac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta}$$

$$(\wedge 2) \quad \frac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$$

$$(\neg 1) \quad \frac{X \vdash \beta, \neg\beta}{X \vdash \alpha}$$

$$(\neg 2) \quad \frac{X, \beta \vdash \alpha | X, \neg\beta \vdash \alpha}{X \vdash \alpha}$$

$$(\forall 1) \quad \frac{X \vdash \forall x(\alpha)}{X \vdash \alpha\frac{t}{x}} \text{ where } \alpha, \tfrac{t}{x} \text{ is collision-free}$$

$$(\forall 2) \quad \frac{X \vdash \alpha\frac{y}{x}}{X \vdash \forall x(\alpha)} \text{ where } y \notin free(X) \cup var(\alpha)$$

$$(=) \quad \frac{X \vdash s = t, \alpha\frac{s}{x}}{X \vdash \alpha\frac{t}{x}} \text{ where } \alpha \text{ is prime}$$