MA 5219 - Logic and Foundations of Mathematics 1
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogic.html`
Homework due in Week 11, Tuesday 29 October 2013.

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Hand in each homework which you want to be checked; 1 mark per each correct starred homework; up to 10 marks in total for homework - there will be more than 10 starred homeworks, so you have several chances to try.

11.1* **Substructures.** Recall that for a logical language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{A}$, the language $\mathcal{LA}$ is the language of all formulas which use besides constants from $\mathcal{L}$ also constants c_a for each a in the domain of $\mathcal{A}$. The diagramme $\mathcal{DA}$ is the set of all true $\mathcal{LA}$-formulas in $\mathcal{A}$ which do not contain any free or bound variable. The elementary diagramme $\mathcal{D}_{el}\mathcal{A}$ is the set of all true $\mathcal{LA}$-sentences in $\mathcal{A}$. Assume that the domain of $\mathcal{A}$ is a subset of the domain of $\mathcal{B}$. Now $\mathcal{A}$ is a substructure of $\mathcal{B}$ iff $\mathcal{B} \models \mathcal{DA}$ and $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ iff $\mathcal{B} \models \mathcal{D}_{el}\mathcal{A}$.

So assume that $\mathcal{A}$ is a 2-dimensional sub space of a given 3-dimensional vector space $\mathcal{B}$ over the finite field $\mathbb{F}_3$ with 3 elements. Is $\mathcal{A}$ a substructure or an elementary substructure of $\mathcal{B}$? Note that the scalar multiplication with 0 is the function mapping all vectors to the zero vector, the scalar multiplication with 1 is the identity mapping and the scalar multiplication with 2 is the mapping $x \mapsto x + x$.

11.2* **Categoricity.** Assume that $\mathcal{L}$ contains infinitely many constants $c_0, c_1, \ldots$ and that $X = \{c_i \neq c_j : i, j \in \mathbb{N} \wedge i \neq j\}$. Is T be the theory of all sentences logically implied by X. Is T $\aleph_0$-categorical? Is T $\aleph_1$-categorical? Justify both answers.

11.3* **Decidability.** Let $\mathcal{L}$ be the logical language with one unary function symbol f, let β be $\forall x\,[f(f(x)) = x]$, let γ be $\forall x, y\,[x = f(x) \wedge y = f(y) \rightarrow x = y]$ and let $T = \{\alpha : \alpha \text{ is a sentence and } \{\beta, \gamma\} \models \alpha\}$. Show that T is decidable.

11.4 Groups. Make a finitely axiomatisable theory T such that (a) every model of T is a group, (b) T has both finite and infinite models and (c) T is decidable.

11.5 Boolean Basis. Let $\mathcal{L}$ be a logical language with the extra symbols $<$ and P and consider the theory T of all sentences implied by the set Y consisting of $\forall x \forall y\,[x < y \vee x = y \vee y < x]$, $\forall x\,[\neg x < x]$, $\forall x \forall y \forall z\,[x < y \wedge y < z \rightarrow x < z]$, $\forall x \exists y \exists z\,[y < x \wedge x < z]$, $\forall x \forall y \exists z\,[x < y \rightarrow x < z \wedge z < y]$, $\forall x \forall y\,[Py \wedge x < y \rightarrow Px]$. Determine a finite set X of sentences which is a Boolean basis for T. That is, X has to satisfy that given any two structures $\mathcal{A}$ and $\mathcal{B}$ of T, either $\mathcal{A}$ and $\mathcal{B}$ are elementary equivalent or there is a sentence α in X such that exactly one of $\mathcal{A}$ and $\mathcal{B}$ makes α true.